

# Classification of sub-Cuntz states

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## Abstract

Let  $\mathcal{O}_n$  denote the Cuntz algebra for  $2 \leq n < \infty$ . With respect to a homogeneous embedding of  $\mathcal{O}_{n^m}$  into  $\mathcal{O}_n$ , an extension of a Cuntz state on  $\mathcal{O}_{n^m}$  to  $\mathcal{O}_n$  is called a sub-Cuntz state, which was introduced by Bratteli and Jorgensen. We show (i) a necessary and sufficient condition of the uniqueness of the extension, (ii) the complete classification of pure sub-Cuntz states up to unitary equivalence of their GNS representations, and (iii) the decomposition formula of a mixing sub-Cuntz state into a convex hull of pure sub-Cuntz states. Invariants of GNS representations of pure sub-Cuntz states are realized as conjugacy classes of nonperiodic homogeneous unit vectors in a tensor-power vector space. It is shown that this state parameterization satisfies both the  $U(n)$ -covariance and the compatibility with a certain tensor product. For proofs of main theorems, matricizations of state parameters and properties of free semigroups are used.

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**Key words.** extension of state, sub-Cuntz state, tensor product formula, matricization, free semigroup.

## 1 Introduction

For a unital  $C^*$ -algebra  $A$  and a unital  $C^*$ -subalgebra  $B$  of  $A$ , any state  $\omega$  on  $B$  has an *extension*  $\tilde{\omega}$  on  $A$ , that is,  $\tilde{\omega}$  is a state on  $A$  which satisfies  $\tilde{\omega}|_B = \omega$  ([15], 2.10.1), but it is not unique in general. In this paper, we completely classify extensions of a certain class of pure states on Cuntz algebras. In consequence, a new class of pure states on Cuntz algebras and the complete set of their invariants are given. In this section, we show our motivation, definitions and main theorems. Proofs will be given after § 3.

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## 1.1 Motivation

In this subsection, we make it clear that our aim of this study against a background of well-known representation theory, and give a short survey.

### 1.1.1 Toward a representation theory of $C^*$ -algebras

According to Kobayashi [41], central problems of representation theory (except interactions with other branches of mathematics) are listed as follows:

**Problem 1** Understanding irreducible representations. Find and classify “smallest” objects. There are the following subproblems:

- Construction of irreducible representations.
- Finding a complete set of *invariants* of representations, so that they can separate different irreducible representations from one another.
- Understanding these invariants.

**Problem 2** Decompose a given representation into irreducible ones. How is a given representation built from “smallest” objects?

**Problem 2-A** Given an irreducible representation  $\tau$  of a subgroup  $G'$ , decompose the induced representation  $\text{Ind}_{G'}^G \tau$  into irreducibles of  $G$ .

**Problem 2-B** Given an irreducible representation  $\pi$  of  $G$ , decompose the restriction  $\pi|_{G'}$  into irreducibles of a subgroup  $G'$ . The formula of the irreducible decomposition in this problem is called a *branching law* (e.g., the decomposition of tensor product representation).

Each problem is more closely explained in the original text. Here a “representation” means a representation of a group  $G$ . We wish to generalize Kobayashi’s problems to the class of algebras which includes group algebras.

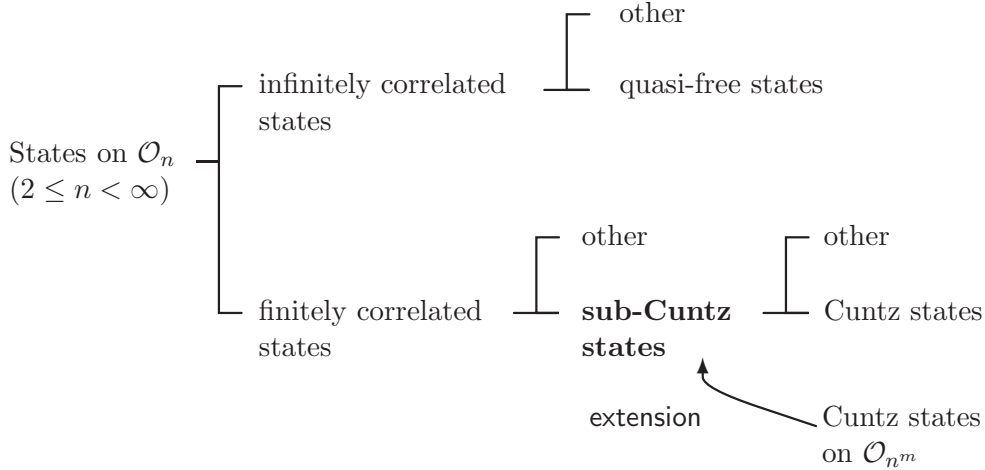
In general, representations of  $C^*$ -algebras do not have unique decomposition (up to unitary equivalence) into sums or integrals of irreducibles [22]. This is a difficulty to consider Problem 2 in the representation theory of  $C^*$ -algebras. However, it does not mean that every irreducible decomposition of a representation of a  $C^*$ -algebra makes no sense. If one chooses a good class of representations, then Problem 2 can be treated satisfactorily. For example, it is known that Cuntz algebras have such good classes of representations.

### 1.1.2 States on Cuntz algebras

By Gel'fand-Naimark-Segal (=GNS) construction, the state theory of a  $C^*$ -algebra  $A$  can be interpreted as the (cyclic) representation theory of  $A$  almost all. Hence, we mainly consider (pure) states instead of (irreducible) representations in this paper. For Cuntz algebras, representations and states have been studied by many authors [3, 4, 6, 7, 8, 9, 10, 19, 20, 21, 26, 27, 28, 42, 46, 48] (see a specific survey in § 1 and § 2 of [17]), but their classifications have not been finished yet. The known most general approach was given in [8]. The set of all states on a Cuntz algebra is divided into two subsets, the set of finitely correlated states and otherwise (= the set of infinitely correlated states) (see § 2.1). A finitely correlated state is characterized by the existence of a finite-dimensional non-trivial  $s_i^*$ -invariant subspace of the GNS representation space [8, 17]. For example, any Cuntz state (see § 1.2) is finitely correlated. There exist both finitely and infinitely correlated vector states of permutative representations [7] (see § 4.2 and Example 4.8).

We illustrate a rough classification of states on  $\mathcal{O}_n$  ( $2 \leq n < \infty$ ) as follows:

**Figure 1.1**



About grounds in Figure 1.1, see the proof of Fact 1.3, Lemma 2.4(i) and Example 4.9. Remark that Figure 1.1 is not true for the case of  $\mathcal{O}_\infty$  (see Proposition C.2). Cuntz states are completely classified pure states with explicit complete invariants, and are used to construct multiplicative isometries ([37], § 3) and  $R$ -matrices ([38], § 3.2) (see also [30, 36]). In this study, we select sub-Cuntz states as a target of complete classification because

they are natural generalizations of Cuntz states. As well as Cuntz states, it is expected that sub-Cuntz states have many applications. Cuntz states and sub-Cuntz states will be explained explicitly in § 1.2. Examples will be shown in § 4.

### 1.1.3 Branching laws of representations of Cuntz algebras

We have mainly studied branching laws of representations of Cuntz algebras according to Kobayashi's Problem 2-B. In [31, 32, 34], branching laws of permutative representations of Cuntz algebras arising from endomorphisms were computed (see also [45]). In [1, 2, 29], representations of fermions were considered as restrictions of representations of  $\mathcal{O}_2$  by a certain embedding of the CAR algebra into  $\mathcal{O}_2$ . By using a certain set of embeddings between Cuntz algebras, we defined a non-symmetric tensor product of representations [33] (see § 1.3.2). We showed the decomposition formula of this tensor product of permutative representations [33, 39]. The set of all unitary equivalence classes of irreducible permutative representations of  $\mathcal{O}_\infty$  is one-to-one correspondence in the set of all equivalence classes of irrational numbers by modular transformations [40]. In this case, finitely and infinitely correlated vector states associated with irreducible components are corresponded to quadratic irrationals and otherwise, respectively.

Their common foundation is the representation theory of Cuntz algebras. Hence its development will be directly reflected in these subjects.

## 1.2 Definition and main theorems

In this subsection, we review the definition of sub-Cuntz state by Bratteli-Jorgensen [7], and show our main theorems for  $\mathcal{O}_n$  ( $2 \leq n < \infty$ ). For the case of  $\mathcal{O}_\infty$ , see Appendix C. Fix  $2 \leq n < \infty$ . Let  $\mathcal{O}_n$  denote the *Cuntz algebra* with Cuntz generators  $s_1, \dots, s_n$ , that is,  $\mathcal{O}_n$  is a  $C^*$ -algebra which is universally generated by  $s_1, \dots, s_n$  which satisfy  $s_i^* s_j = \delta_{ij} I$  for  $i, j = 1, \dots, n$  and  $\sum_{i=1}^n s_i s_i^* = I$  [12].

We review Cuntz state before sub-Cuntz state. For any complex unit vector  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ , a state  $\omega_z$  on  $\mathcal{O}_n$  which satisfies

$$\omega_z(s_j) = \overline{z_j} \quad \text{for all } j = 1, \dots, n, \quad (1.1)$$

exists uniquely and is pure where  $\overline{z_i}$  denotes the complex conjugate of  $z_i$ . The state  $\omega_z$  is called the *Cuntz state* by  $z$  [6, 7, 10]. GNS representations by  $\omega_z$  and  $\omega_y$  are unitarily equivalent if and only if  $z = y$  (see Appendix B). About equivalent definitions, see the case of  $m = 1$  in Theorem 2.3.

For  $m \geq 1$ , let  $\mathcal{V}_{n,m}$  denote the Hilbert space with an orthonormal basis  $\{e_J : J \in \{1, \dots, n\}^m\}$ , that is,  $\mathcal{V}_{n,m} = \ell^2(\{1, \dots, n\}^m)$ . Let  $(\mathcal{V}_{n,m})_1 := \{z \in \mathcal{V}_{n,m} : \|z\| = 1\}$ .

**Definition 1.2** For  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ ,  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  if  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies the following equations:

$$\omega(s_J) = \overline{z_J} \quad \text{for all } J \in \{1, \dots, n\}^m \quad (1.2)$$

where  $s_J := s_{j_1} \cdots s_{j_m}$  when  $J = (j_1, \dots, j_m)$ , and  $\overline{z_J}$  denotes the complex conjugate of  $z_J$ . In this case,  $\omega$  is called a sub-Cuntz state of order  $m$ .

This definition is equivalent to the original in [7] (see Theorem 2.3).

**Fact 1.3** (Existence) For any  $z \in (\mathcal{V}_{n,m})_1$ , a sub-Cuntz state by  $z$  exists.

*Proof.* Fix a bijection  $f : \{1, \dots, n^m\} \cong \{1, \dots, n\}^m$ . Let  $t_1, \dots, t_{n^m}$  denote the Cuntz generators of  $\mathcal{O}_{n^m}$ . Define the embedding  $\hat{f}$  of  $\mathcal{O}_{n^m}$  into  $\mathcal{O}_n$  by  $\hat{f}(t_i) := s_{j_1} \cdots s_{j_m}$  for  $i \in \{1, \dots, n^m\}$  when  $f(i) = (j_1, \dots, j_m)$ . We identify  $\mathcal{O}_{n^m}$  with  $\hat{f}(\mathcal{O}_{n^m})$  here. By definition,  $\omega$  is a sub-Cuntz state by  $z$  if and only if  $\omega$  is an extension of the Cuntz state  $\omega_{\hat{z}}$  on  $\mathcal{O}_{n^m}$  to  $\mathcal{O}_n$  where  $\hat{z} := (z_{f(i)})_{i=1}^{n^m} \in (\mathbb{C}^{n^m})_1$ . Since an extension of  $\omega_{\hat{z}}$  always exists, the statement holds.  $\blacksquare$

From the proof of Fact 1.3, a sub-Cuntz state is regarded as an extension of a Cuntz state. Such an extension always exists but it is not always unique. We show a necessary and sufficient condition of its uniqueness as follows. Here we identify  $\mathcal{V}_{n,m}$  with  $(\mathcal{V}_{n,1})^{\otimes m}$  by the correspondence between bases  $e_J \mapsto e_{j_1} \otimes \cdots \otimes e_{j_m}$  for  $J = (j_1, \dots, j_m) \in \{1, \dots, n\}^m$ . From this identification, we obtain  $\mathcal{V}_{n,m} \otimes \mathcal{V}_{n,l} = \mathcal{V}_{n,m+l}$  for any  $m, l \geq 1$ . Then the following hold.

**Theorem 1.4** Let  $\omega$  be a sub-Cuntz state on  $\mathcal{O}_n$  by  $z \in (\mathcal{V}_{n,m})_1$ .

- (i) (Uniqueness)  $\omega$  is unique if and only if  $z$  is nonperiodic (or primitive [47]), that is,  $z = x^{\otimes p}$  for some  $x$  implies  $p = 1$ . In this case, we write  $\tilde{\omega}_z$  as  $\omega$ .
- (ii) If  $z$  is nonperiodic, then  $\tilde{\omega}_z$  is pure.

(iii) If  $z$  is nonperiodic, then

$$\tilde{\omega}_z(s_J s_K^*) = \begin{cases} \overline{z_J} z_K & \text{when } |J|, |K| \in m\mathbb{Z}_{\geq 0}, \\ 0 & \text{when } |J| - |K| \notin m\mathbb{Z}, \\ \overline{z_{J_1}} z_{K_1} \sum_{|L|=m-|J_2|} \overline{z_{J_2 L}} z_{K_2 L} & \text{otherwise} \end{cases} \quad (1.3)$$

for  $J, K \in \bigcup_{a \geq 1} \{1, \dots, n\}^a \cup \{\emptyset\}$  where  $|J|$  denotes the word length of  $J$ ,  $JK$  denotes the concatenation of  $J$  and  $K$ ,  $s_\emptyset := I$ ,  $z_\emptyset := 1$  and  $z_J := z_{J^{(1)}} \cdots z_{J^{(l)}}$  when  $J = J^{(1)} \cdots J^{(l)}$  and  $|J^{(i)}| = m$  for  $i = 1, \dots, l$ . In the case of “otherwise” in (1.3),  $J$  and  $K$  satisfy  $J = J_1 J_2$  and  $K = K_1 K_2$  such that  $|J_1|, |K_1| \in m\mathbb{Z}_{\geq 0}$  and  $1 \leq |J_2| = |K_2| \leq m - 1$ .

From Theorem 1.4(i), if  $z$  is periodic (= not nonperiodic), then  $\omega$  is not unique. In this case, all possibilities of sub-Cuntz states by  $z$  are explicitly given as follows.

**Theorem 1.5** (Decomposition) *Let  $p \geq 2$  and  $z := x^{\otimes p}$  for a nonperiodic element  $x \in (\mathcal{V}_{n,m'})_1$ . If  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$ , then there exists  $a = (a_1, \dots, a_p)$  in  $\Delta_{p-1} := \{(b_1, \dots, b_p) \in \mathbb{R}^p : \sum b_j = 1, b_i \geq 0 \text{ for all } i\}$  such that  $\omega$  has the following form:*

$$\omega = \sum_{j=1}^p a_j \omega_j \quad (1.4)$$

where  $\omega_j$  denotes the pure sub-Cuntz state by  $e^{2\pi j \sqrt{-1}/p} x$ . In (1.4),  $(a_1, \dots, a_p)$  is unique.

From Theorem 1.5, we see that a sub-Cuntz state by  $z$  may be pure even if  $z$  is periodic. By combining Theorem 1.4(ii) and Theorem 1.5, we obtain the following necessary and sufficient condition of the purity of sub-Cuntz state.

**Corollary 1.6** (Purity) *For a sub-Cuntz state  $\omega$  by  $z \in (\mathcal{V}_{n,m})_1$ ,  $\omega$  is pure if and only if  $\omega = \tilde{\omega}_x$  for some nonperiodic element  $x \in (\mathcal{V}_{n,m'})_1$ . In this case,  $z = x^{\otimes p}$  for some  $p \geq 1$ .*

*Proof.* ( $\Rightarrow$ ) Assume that  $\omega$  is pure. If  $z$  is nonperiodic, then let  $x := z$ . If  $z = v^{\otimes p}$  for some nonperiodic element  $v$  and  $p \geq 2$ , then  $\omega = \sum_{j=1}^p a_j \omega_j$

from Theorem 1.5 where  $\omega_j$  denotes the pure sub-Cuntz state by  $e^{2\pi j\sqrt{-1}/p}v$ . Since  $\omega$  is pure and  $\omega_i \neq \omega_j$  when  $i \neq j$ , there must exist  $j$  such that  $a_j = 1$  and  $\omega = \omega_j$ . Let  $x := e^{2\pi j\sqrt{-1}/p}v$ . Then the statement holds.

( $\Leftarrow$ ) From Theorem 1.4(ii), the statement holds.

From the above proofs, we see that  $z = x^{\otimes p}$  for  $p \geq 1$ . ■

Next, we consider an equivalence of sub-Cuntz states.

**Theorem 1.7 (Equivalence)** *For  $z, y \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1$ , assume that both  $z$  and  $y$  are nonperiodic. Then the following are equivalent:*

- (i) *GNS representations by  $\tilde{\omega}_z$  and  $\tilde{\omega}_y$  are unitarily equivalent. In this case, we write  $\tilde{\omega}_z \sim \tilde{\omega}_y$ .*
- (ii) (a)  $z = y$ , or  
(b)  $z = x_1 \otimes x_2$  and  $y = x_2 \otimes x_1$  for some  $x_1, x_2 \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1$ .

*In these cases,  $z$  and  $y$  are said to be conjugate ([47], § 1.3), and we write  $z \sim y$ .*

Assume that both  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$  are nonperiodic. If  $m \neq l$ , then  $z \not\sim y$ . Hence  $\tilde{\omega}_z \not\sim \tilde{\omega}_y$  from Theorem 1.7. From Theorem 1.5, two sub-Cuntz states of different orders may be equivalent.

**Remark 1.8** (i) By definition,  $\omega$  is a sub-Cuntz state of order 1 if and only if  $\omega$  is a Cuntz state. From Theorem 1.7, any Cuntz state is not equivalent to any sub-Cuntz state by a nonperiodic parameter  $z \in (\mathcal{V}_{n,m})_1$  for  $m \geq 2$ .

(ii) The restriction of any sub-Cuntz state on  $\mathcal{O}_n$  by  $z$  on the UHF subalgebra  $UHF_n := C^*\{s_J s_K^* \in \mathcal{O}_n : |J| = |K|\}$  of  $\mathcal{O}_n$  is always uniquely defined by  $z$  from Theorem 1.4(iii) (see also [7], Proposition 5.1).

(iii) For  $\omega$  in (1.4), we have the unique irreducible decomposition of the GNS representation  $\pi$  by  $\omega$  as follows:

$$\pi = \bigoplus_{j=1}^p \hat{a}_j \pi_j, \quad \hat{a}_j := \begin{cases} 1 & (a_j > 0), \\ 0 & (a_j = 0) \end{cases} \quad (1.5)$$

where  $\pi_j$  denotes the (irreducible) GNS representation by  $\omega_j$ , and  $\hat{a}_j$  means the multiplicity coefficient of  $\pi_j$  in  $\pi$ . By Theorem 1.7,  $\pi_i \not\sim \pi_j$  when  $i \neq j$  because  $e^{2\pi i\sqrt{-1}/p}x \not\sim e^{2\pi j\sqrt{-1}/p}x$  when  $i \neq j$ . Hence  $\pi$

is multiplicity free. In consequence, the GNS representation by any sub-Cuntz state is multiplicity free, and the class of GNS representations by sub-Cuntz states is closed with respect to the irreducible decomposition.

- (iv) We can verify that  $\sim$  in Theorem 1.7(ii) is an equivalence relation. Let  $\approx$  denote the equivalence relation on  $\mathcal{V}_{n,m} = (\mathbb{C}^n)^{\otimes m}$  by the action of the cyclic group  $\mathbb{Z}/m\mathbb{Z}$  with respect to permutations of tensor components. Then  $\sim$  does not coincide with  $\approx$ . For example, define three vectors in  $(\mathcal{V}_{n,3})_1$ ,  $n \geq 3$  by

$$\begin{cases} z^{(1)} := e_1 \otimes \frac{e_1 \otimes e_2 + e_2 \otimes e_3}{\sqrt{2}}, \\ z^{(2)} := \frac{e_1 \otimes e_2 + e_2 \otimes e_3}{\sqrt{2}} \otimes e_1, \\ z^{(3)} := \frac{e_2 \otimes e_1 \otimes e_1 + e_3 \otimes e_1 \otimes e_2}{\sqrt{2}}. \end{cases} \quad (1.6)$$

Then  $z^{(1)} \approx z^{(2)} \approx z^{(3)}$ , but  $z^{(1)} \sim z^{(2)} \not\sim z^{(3)}$ .

### 1.3 Naturalities of state parameterization

In Theorem 1.4, we introduced a parametrization of pure sub-Cuntz states:

$$z \longmapsto \tilde{\omega}_z. \quad (1.7)$$

In this subsection, we show how natural this parameterization is.

#### 1.3.1 $U(n)$ -covariance

For convenience, we introduce some symbols as follows.

**Corollary 1.9** *Define*

$$\mathfrak{N}_n := \{z \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1 : z \text{ is nonperiodic}\},$$

$$\mathfrak{I}_n := \{z \in \bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1 : z \text{ is indecomposable}\},$$

$$\mathcal{P}_n : \text{ the set of all pure states on } \mathcal{O}_n,$$

$$\mathcal{P}_{n,\text{sub}} : \text{ the set of all pure sub-Cuntz states on } \mathcal{O}_n,$$

$$\text{Spec} \mathcal{O}_n : \text{ the set of all unitary equivalence classes of irreducible representations of } \mathcal{O}_n$$



where  $z$  is said to be indecomposable if  $z$  can not be written as  $z_1 \otimes z_2$  for any  $z_1, z_2$ . Then the following hold:

- (i) The map  $q : \mathfrak{N}_n \rightarrow \mathcal{P}_{n,sub}$ ;  $q(z) := \tilde{\omega}_z$ , is bijective.
- (ii) The map  $r : \mathfrak{I}_n \rightarrow \text{Spec} \mathcal{O}_n$ ;  $r(z) := [\pi_z]$ , is injective where  $[\pi_z]$  denotes the unitary equivalence class of the GNS representation  $\pi_z$  by  $\tilde{\omega}_z$ .

*Proof.* (i) From Corollary 1.6, the statement holds.

(ii) From Theorem 1.7, if both  $z$  and  $y$  are indecomposable, then  $\tilde{\omega}_z \sim \tilde{\omega}_y$  if and only if  $z = y$ . From this, the statement holds.  $\blacksquare$

From Theorem 1.7,  $\mathfrak{N}_n/\sim \cong \mathcal{P}_{n,sub}/\sim$  and  $\mathfrak{I}_n/\sim = \mathfrak{I}_n$ . The parameter set  $\mathfrak{I}_n$  can be regarded as the set of non-commutative homogeneous irreducible polynomials in  $n$ -variables with the norm 1 [43].

We show a naturality of the parameterization in Corollary 1.9 with respect to the standard unitary group action  $\alpha$  on  $\mathcal{O}_n$ , which is defined as

$$\alpha_g(s_i) := \sum_{j=1}^n g_{ji} s_j \quad (i = 1, \dots, n, g = (g_{ij}) \in U(n)). \quad (1.8)$$

Define the dual action  $\alpha^*$  of  $\alpha$  on the dual  $\mathcal{O}_n^*$  of  $\mathcal{O}_n$  by  $\alpha_g^*(f) := f \circ \alpha_g$  for  $f \in \mathcal{O}_n^*$  and  $g \in U(n)$ . Especially,  $\alpha_g^*(\mathcal{P}_n) = \mathcal{P}_n$  for all  $g$ .

Let  $\gamma$  denote the standard action of  $U(n)$  on  $\mathcal{V}_{n,1} = \mathbb{C}^n$ , that is,

$$\gamma_g e_i := \sum_{j=1}^n g_{ji} e_j \quad (i = 1, \dots, n, g \in U(n)). \quad (1.9)$$

Since  $\gamma_g^{\otimes m} := (\gamma_g)^{\otimes m}$  is a unitary,  $\gamma_g^{\otimes m}((\mathcal{V}_{n,m})_1) = (\mathcal{V}_{n,m})_1$  for all  $g \in U(n)$ . Define the action  $\Gamma$  of  $U(n)$  on  $\bigcup_{m \geq 1} (\mathcal{V}_{n,m})_1$  by  $\Gamma_g z := \gamma_g^{\otimes m} z$  when  $z \in (\mathcal{V}_{n,m})_1$ . Remark that if  $z$  is nonperiodic (*resp.* indecomposable), then  $\Gamma_g z$  is also nonperiodic (*resp.* indecomposable) for any  $g \in U(n)$ .

**Proposition 1.10** ( *$U(n)$ -covariance*) Let  $\mathfrak{N}_n$  be as in Corollary 1.9. For any  $g \in U(n)$  and  $z \in \mathfrak{N}_n$ ,

$$\alpha_g^*(\tilde{\omega}_z) = \tilde{\omega}_{\Gamma_g z}. \quad (1.10)$$

That is, the parameterization  $z \mapsto \tilde{\omega}_z$  is  $U(n)$ -covariant.

*Proof.* Assume  $z \in (\mathcal{V}_{n,m})_1 \cap \mathfrak{N}_n$ . By definition, we can verify  $\alpha_g^*(\tilde{\omega}_z)(s_J) = \overline{(\gamma_g^{\otimes m} z)_J} = \overline{(\Gamma_g z)_J}$  for all  $J \in \{1, \dots, n\}^m$  and  $g \in U(n)$ . Since  $\Gamma_g z$  is non-periodic,  $\alpha_g^*(\tilde{\omega}_z)$  coincides with  $\tilde{\omega}_{\Gamma_g z}$  from Theorem 1.4(i).  $\blacksquare$

In other words,  $q$  in Corollary 1.9(i) is an isomorphism between two dynamical systems  $(\mathfrak{N}_n, \Gamma, U(n))$  and  $(\mathcal{P}_{n,sub}, \alpha^*, U(n))$ .

### 1.3.2 Compatibility with $\varphi$ -tensor product

In [33], we introduced a non-symmetric tensor product of states on Cuntz algebras. In this subsection, we show tensor product formulas of sub-Cuntz states.

We review definitions in [33]. Let  $s_1^{(n)}, \dots, s_n^{(n)}$  denote Cuntz generators of  $\mathcal{O}_n$ . For  $2 \leq n, n' < \infty$ , define the unital  $*$ -embedding  $\varphi_{n,n'}$  of  $\mathcal{O}_{nn'}$  into  $\mathcal{O}_n \otimes \mathcal{O}_{n'}$  by  $\varphi_{n,n'}(s_{n'(i-1)+j}^{(nn')}) := s_i^{(n)} \otimes s_j^{(n')}$  for  $i = 1, \dots, n, j = 1, \dots, n'$ . Let  $\mathcal{S}_n$  denote the set of all states on  $\mathcal{O}_n$ . For  $(\omega_1, \omega_2) \in \mathcal{S}_n \times \mathcal{S}_{n'}$ , the  $\varphi$ -tensor product  $\omega_1 \otimes_\varphi \omega_2 \in \mathcal{S}_{nn'}$  is defined by

$$\omega_1 \otimes_\varphi \omega_2 := (\omega_1 \otimes \omega_2) \circ \varphi_{n,n'}. \quad (1.11)$$

Then  $\otimes_\varphi$  is associative. Hence the set  $\bigcup_{n \geq 2} \mathcal{S}_n$  is a semigroup with the product  $\otimes_\varphi$ . Furthermore, the following holds.

**Proposition 1.11** *The  $\varphi$ -tensor product of any two sub-Cuntz states is also a sub-Cuntz state, that is, the set of all sub-Cuntz states is closed with respect to  $\otimes_\varphi$ .*

For  $J = (j_1, \dots, j_m) \in \{1, \dots, n\}^m$  and  $K = (k_1, \dots, k_m) \in \{1, \dots, n'\}^m$ , define  $J \boxtimes K = (l_1, \dots, l_m) \in \{1, \dots, nn'\}^m$  by  $l_t := n'(j_t - 1) + k_t$  for  $t = 1, \dots, m$ . For  $z = \sum z_J e_J \in \mathcal{V}_{n,m}$  and  $y = \sum y_K e_K \in \mathcal{V}_{n',m}$ , define  $z \boxtimes y \in \mathcal{V}_{nn',m}$  by

$$z \boxtimes y := \sum_{L \in \{1, \dots, nn'\}^m} (z \boxtimes y)_L e_L, \quad (z \boxtimes y)_L := z_J y_K \quad (1.12)$$

where  $J \in \{1, \dots, n\}^m$  and  $K \in \{1, \dots, n'\}^m$  are uniquely defined as  $J \boxtimes K = L$ . By definition,  $\|z \boxtimes y\| = \|z\| \cdot \|y\|$ . If  $\|z\| = \|y\| = 1$ , then  $\|z \boxtimes y\| = 1$ . Remark that for any  $z, y \in \mathcal{V}_{n,m}$ ,  $z \otimes y \in \mathcal{V}_{n,2m}$  and  $z \boxtimes y \in \mathcal{V}_{n^2,m}$ . Clearly,  $\mathcal{V}_{n,2m} \cong \mathcal{V}_{n^2,m}$ , but we distinguish  $\otimes$  from  $\boxtimes$  here.

In addition to  $\boxtimes$ , we define a new operation. For  $z = \sum z_J e_J \in \mathcal{V}_{n,m}$  and  $y = \sum y_K e_K \in \mathcal{V}_{n',l}$ , define

$$z * y := z^{\otimes \alpha} \boxtimes y^{\otimes \beta} \in \mathcal{V}_{nn',d} \quad (1.13)$$

where  $d, \alpha, \beta \geq 1$  are uniquely chosen such that  $d := \alpha m = \beta l$  is the least common multiple of  $m$  and  $l$ . Especially, if  $m = l$ , then  $\alpha = \beta = 1$ ,  $d = m$  and  $z * y = z \boxtimes y$ . If  $\|z\| = \|y\| = 1$ , then  $\|z * y\| = 1$ . About examples of these operations, see [33].

**Proposition 1.12** *Assume that both  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n',l})_1$  are non-periodic. Then the following hold:*

- (i)  $z * y$  is nonperiodic.
- (ii) (Tensor product formula)  $\tilde{\omega}_z \otimes_{\varphi} \tilde{\omega}_y = \tilde{\omega}_{z*y}$ . Especially, if  $m = l$ , then  $\tilde{\omega}_z \otimes_{\varphi} \tilde{\omega}_y = \tilde{\omega}_{z \boxtimes y}$ .

Let  $\omega_z$  denote the Cuntz state on  $\mathcal{O}_n$  by  $z \in (\mathbb{C}^n)_1$ . As special cases of Proposition 1.10 and Proposition 1.12(ii), the following hold:

$$\alpha_g^*(\omega_z) = \omega_{gz}, \quad \omega_z \otimes_{\varphi} \omega_y = \omega_{z \boxtimes y} \quad (1.14)$$

for any  $z \in (\mathbb{C}^n)_1$ ,  $y \in (\mathbb{C}^{n'})_1$  and  $g \in U(n)$  where  $gz = \gamma_g z$ .

From Proposition 1.12(ii), the operation  $*$  is associative because  $\otimes_{\varphi}$  is associative. Proposition 1.12(i) means that  $\mathfrak{N}_* := \bigcup_{n \geq 2} \mathfrak{N}_n$  is a semigroup with the product  $*$ . Furthermore, the following holds from Corollary 1.6.

**Corollary 1.13** *For  $q$  and  $\mathcal{P}_{n,sub}$  in Corollary 1.9, the set  $\mathcal{P}_{*,sub} := \bigcup_{n \geq 2} \mathcal{P}_{n,sub}$  is a semigroup with the product  $\otimes_{\varphi}$ , and  $q$  can be extended to an isomorphism between  $(\mathfrak{N}_*, *)$  onto  $(\mathcal{P}_{*,sub}, \otimes_{\varphi})$ .*

Corollary 1.13 means the second naturality of the state parametrization  $q$ .

The paper is organized as follows: In § 2, we will review known results and prepare tools to prove main theorems. In § 2.2, a matricization of state parameter will be introduced. In § 3, we will prove main theorems. In § 4, we will show examples. In § 4.1, we will show sub-Cuntz states of order 2. In § 4.2, sub-Cuntz states associated with permutative representations will be explained. In § 4.3, examples of non-sub-Cuntz states will be shown.

## 2 Preparations

### 2.1 Finitely correlated states on $\mathcal{O}_n$

We start from general properties of extensions of states.

**Proposition 2.1** ([15], 2.10.1) *Let  $A$  be a  $C^*$ -algebra with unit  $I$ , and  $B$  a  $C^*$ -subalgebra of  $A$  such that  $I \in B$ . Then the following hold:*

- (i) Every state on  $B$  can be extended to a state on  $A$ .
- (ii) Every pure state on  $B$  can be extended to a pure state on  $A$ . Especially, if its extension is unique, then it is pure.

The existence of sub-Cuntz state is assured by Proposition 2.1(i). If it is unique, then its purity is assured by Proposition 2.1(ii).

**Definition 2.2** ([6, 8]) *A state  $\omega$  on  $\mathcal{O}_n$  ( $2 \leq n \leq \infty$ ) is said to be finitely correlated if  $\dim \mathcal{K} < \infty$  where  $\mathcal{K} := \overline{\text{Lin}\langle\{\pi(s_J)^*\Omega \in \mathcal{H} : J\}\rangle}$  and  $(\mathcal{H}, \pi, \Omega)$  denotes the GNS representation by  $\omega$ . If not,  $\omega$  is said to be infinitely correlated.*

Next, we show equivalent definitions of sub-Cuntz state as follows.

**Theorem 2.3** *Fix  $m \geq 1$ . Let  $\omega$  be a state on  $\mathcal{O}_n$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ . For  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ , the following conditions are equivalent:*

- (i)  $\omega$  is a sub-Cuntz state by  $z$ .
- (ii)  $\Omega = \pi(s(z))\Omega$  where  $s(z) := \sum z_J s_J$ .
- (iii)  $\pi(s_J)^*\Omega = z_J \Omega$  for all  $J \in \{1, \dots, n\}^m$ .

*Proof.* From Proposition 5.1 of [7], (ii) and (iii) are equivalent. By the definition of  $(\mathcal{H}, \pi, \Omega)$ , (iii) implies (i). From (i), we have  $\sum z_J \omega(s_J) = 1$ . This implies  $\|\Omega - \sum z_J \pi(s_J)\| = 0$ . Hence (ii) holds.  $\blacksquare$

In Definition 5.7 of [17], a cyclic representation of  $\mathcal{O}_n$  which satisfies equations in Theorem 2.3(iii) with  $m = 1$  is called a *generic representation*.

**Lemma 2.4** (i) *When  $n < \infty$ , any sub-Cuntz state on  $\mathcal{O}_n$  is finitely correlated.*

- (ii) *If  $\omega$  is a sub-Cuntz state with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ , then  $\mathcal{H} = \overline{\text{Lin}\langle\{\pi(s_J)\Omega : J\}\rangle}$ .*

*Proof.* Assume that  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ .  
(i) By Theorem 2.3(iii),  $\dim \text{Lin}\langle\{\pi(s_J^*)\Omega : J\}\rangle \leq \sum_{l=0}^{m-1} \dim \text{Lin}\langle\{\pi(s_J^*)\Omega : |J| = l\}\rangle \leq \sum_{l=0}^{m-1} n^l < \infty$ .  
(ii) Since  $\mathcal{O}_n$  is spanned by  $\{s_J s_K^* : J, K\}$ ,  $\mathcal{H}$  is spanned by  $\{\pi(s_J s_K^*)\Omega : J, K\}$ . From Theorem 2.3(ii),  $\pi(s_J s_K^*)\Omega = \pi(s_J s_K^*(s(z))^l)\Omega$  for any  $l \geq 1$ . Therefore  $\pi(s_J s_K^*)\Omega \in \text{Lin}\langle\{\pi(s_{J'})\Omega : J'\rangle\rangle$  for any  $J, K$ . This implies the statement.  $\blacksquare$

Remark that Lemma 2.4(i) does not hold for  $\mathcal{O}_\infty$  (see Proposition C.2).

## 2.2 Matricization of state parameter

Assume  $m \geq 2$ . In this subsection, we introduce operators associated with an element  $z \in \mathcal{V}_{n,m}$ .

For  $x \in \mathcal{V}_{n,a}$  and  $y \in \mathcal{V}_{n,b}$  with  $a, b \geq 1$ , define the operator  $x \otimes y^*$  from  $\mathcal{V}_{n,b}$  to  $\mathcal{V}_{n,a}$  by  $(x \otimes y^*)v := \langle y|v \rangle x$  for  $v \in \mathcal{V}_{n,b}$ . We generalize this as follows. For  $z = \sum z_M e_M \in \mathcal{V}_{n,m}$  and  $1 \leq a \leq m-1$ , define the operator  $T_a(z)$  from  $\mathcal{V}_{m,a}$  to  $\mathcal{V}_{n,m-a}$  by

$$T_a(z)e_K := \sum_{|J|=m-a} z_{JK} e_J \quad (K \in \{1, \dots, n\}^a). \quad (2.1)$$

In other words,  $T_a(z)v = \sum_{|J|=m-a} \langle \bar{z} | e_J \otimes v \rangle e_J$  for  $v \in \mathcal{V}_{n,a}$ , or  $T_a(z) = \sum_{J,K} z_{JK} e_J \otimes e_K^*$  where  $\bar{z} := \sum \bar{z}_M e_M$ . The operator  $T_a(z)$  is called the *matricizing* (matricization) [18], *unfolding* [44] or *flattening* [49] of a tensor  $z \in \mathcal{V}_{n,m} = (\mathbb{C}^n)^{\otimes m}$ . Especially,  $T_a(x \otimes y) = x \otimes \bar{y}^*$  for any  $x \in \mathcal{V}_{n,m-a}$  and  $y \in \mathcal{V}_{n,a}$ .

In the case of  $m = 2$ ,  $T_1(z)$  is identified with the matrix representation  $(z_{ij}) \in M_n(\mathbb{C})$  of a 2-tensor  $z = \sum_{i,j=1}^n z_{ij} e_{ij} \in \mathcal{V}_{n,2}$  by definition. In general case, by the identification  $\text{Hom}_{\mathbb{C}}(\mathcal{V}_{n,a}, \mathcal{V}_{n,m-a})$  with the set  $M_{n^{m-a}, n^a}(\mathbb{C})$  of all  $n^{m-a} \times n^a$  matrices,  $T_a$  is regarded as the following mapping:

$$(\mathbb{C}^n)^{\otimes m} = \mathcal{V}_{n,m} \ni z \mapsto T_a(z) \in M_{n^{m-a}, n^a}(\mathbb{C}). \quad (2.2)$$

In order to show properties of  $T_a(z)$ , we review operator norms as follows. Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces. For a bounded linear operator  $A$  from  $\mathcal{H}$  to  $\mathcal{K}$ , the *uniform norm* and the *Hilbert-Schmidt norm* of  $A$  are defined as  $\|A\| := \sup_{x \in \mathcal{H}, \|x\|=1} \|Ax\|$  and  $\|A\|_2 := (\text{tr} A^* A)^{1/2}$ , respectively [5, 16, 23]. Then  $\|A\| \leq \|A\|_2$ . Furthermore,  $\|A\| = \|A\|_2 \neq 0$  if and only if there exist  $y \in \mathcal{H}$  and  $x \in \mathcal{K}$  such that  $x, y \neq 0$  and  $A = x \otimes y^*$ .

**Fact 2.5** *Let  $z \in \mathcal{V}_{n,m}$ .*

- (i) *For  $1 \leq a \leq m-1$ ,  $\|T_a(z)\| \leq \|z\|$ .*
- (ii) *If  $z \neq 0$ , then  $\|T_a(z)\| = \|z\|$  if and only if  $z = x \otimes y$  for some  $x \in \mathcal{V}_{n,m-a}$  and  $y \in \mathcal{V}_{n,a}$ .*

*Proof.* (i) From the inequality of norms and (2.1),

$$\|T_a(z)\| \leq \|T_a(z)\|_2 = \left\{ \sum_{|K|=a} \sum_{|J|=m-a} |z_{JK}|^2 \right\}^{1/2} = \left\{ \sum_{|M|=m} |z_M|^2 \right\}^{1/2} = \|z\|. \quad (2.3)$$

(ii) Assume  $\|T_a(z)\| = \|z\|$ . From the proof of (i),  $\|T_a(z)\| = \|z\| = \|T_a(z)\|_2$ . Hence  $T_a(z) = x \otimes w^*$  for some  $x \in \mathcal{V}_{n,m-a}$  and  $w \in \mathcal{V}_{n,a}$ . This implies  $z_{JK} = x_J \overline{w_K} = (x \otimes \overline{w})_{JK}$  for  $J \in \{1, \dots, n\}^{m-a}$  and  $K \in \{1, \dots, n\}^a$ . By taking  $y := \overline{w}$ ,  $z = x \otimes y$ . The inverse direction holds from (2.1).  $\blacksquare$

The following is one of key lemmas to prove main theorems.

**Lemma 2.6** *Let  $m \geq 2$ .*

- (i) *For  $X, Y \in (\mathcal{V}_{n,m})_1$ , if there exists a nonzero vector  $v \in \mathcal{V}_{n,a}$  for some  $1 \leq a \leq m-1$  which satisfies*

$$v = c T_{m-a}(\overline{X}) T_a(Y) v \quad (2.4)$$

*for some  $c \in U(1) := \{c' \in \mathbb{C} : |c'| = 1\}$ , then there exist  $x_1 \in (\mathcal{V}_{n,a})_1$  and  $x_2 \in (\mathcal{V}_{n,m-a})_1$  such that  $X = x_1 \otimes x_2$  and  $Y = \overline{c} x_2 \otimes x_1$ .*

- (ii) *In addition to (i), if  $X = Y$ , then  $c = 1$  and  $X$  is periodic.*

*Proof.* (i) From Fact 2.5(i),  $\|T_b(z)\| \leq \|z\| = 1$  for  $z = X, Y$  and  $b = a, m-a$ . Since  $v \neq 0$ , we obtain  $\|T_a(Y)\| = \|T_{m-a}(X)\| = \|T_{m-a}(\overline{X})\| = 1 = \|X\| = \|Y\|$  from (2.4). From these and Fact 2.5(ii),

$$X = x_1 \otimes x_2, \quad Y = x'_1 \otimes x'_2 \quad (2.5)$$

for some  $x_1, x'_2 \in (\mathcal{V}_{n,a})_1$  and  $x_2, x'_1 \in (\mathcal{V}_{n,m-a})_1$ . By substituting (2.5) into (2.4),

$$v = c \overline{x_1} \langle x_2 | x'_1 \rangle \langle \overline{x'_2} | v \rangle. \quad (2.6)$$

From (2.6),  $\langle \overline{x'_2} | v \rangle = c \langle \overline{x'_2} | \overline{x_1} \rangle \langle x_2 | x'_1 \rangle \langle \overline{x'_2} | v \rangle$ . By  $v \neq 0$  and (2.6),  $\langle \overline{x'_2} | v \rangle \neq 0$ . Hence  $1 = c \langle \overline{x'_2} | \overline{x_1} \rangle \langle x_2 | x'_1 \rangle = \langle \overline{c} x_2 \otimes x_1 | x'_1 \otimes x'_2 \rangle$ . From Lemma A.1,  $x'_1 \otimes x'_2 = \overline{c} x_2 \otimes x_1$ . From this and (2.5), the statement holds.

(ii) From (i),  $\overline{c} x_2 \otimes x_1 = Y = X = x_1 \otimes x_2$ . Applying Corollary A.6(i) to this, the statement holds.  $\blacksquare$

### 2.3 Reduction of problems

For a given sub-Cuntz state  $\omega$  of order  $m \geq 2$ , Definition 1.2 determines only special values  $\{\omega(s_J) : |J| = m\}$ , but not all values  $\{\omega(s_J s_K^*) : J, K\}$ . From Theorem 2.3, we can reduce the uniqueness problem of  $\omega$  to the problem to determine the smaller set  $\{\omega(s_J) : 1 \leq |J| \leq m-1\}$ .

**Lemma 2.7** For  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$ , we introduce the following assumptions:

**Assumption E:** “ $\mathcal{O}_n$  acts on a Hilbert space  $\mathcal{H}$  with two unit vectors  $\Omega_z$  and  $\Omega_y$  which satisfy

$$s(z)\Omega_z = \Omega_z, \quad s(y)\Omega_y = \Omega_y \quad (2.7)$$

where  $s(z) := \sum z_J s_J \in \mathcal{O}_n$  for  $z = \sum z_J e_J$ .”

**Assumption EC:** Assumption E with the cyclicity of both  $\Omega_z$  and  $\Omega_y$ .

Assume Assumption E for  $z$  and  $y$ . Define the linear functional  $\omega_{z,y}$  on  $\mathcal{O}_n$  by

$$\omega_{z,y} := \langle \Omega_z | (\cdot) \Omega_y \rangle. \quad (2.8)$$

Then the following hold.

- (i) For  $J, K \in \bigcup_{k \geq 0} \{1, \dots, n\}^k$ , assume that  $J = J_1 J_2, K = K_1 K_2, |J_1| = ma, |K_1| = lb$  for some  $a, b \geq 0$ , and  $|J_2| = i, |K_2| = i', 0 \leq i \leq m-1$  and  $0 \leq i' \leq l-1$ .

(a) If  $(i, i') = (0, 0)$ , then  $\omega_{z,y}(s_J s_K^*) = \overline{z_J} y_K \omega_{z,y}(I)$ .

(b) If  $(i, i') \neq (0, 0)$  and  $\alpha m - i = \beta l - i'$  for some  $\alpha, \beta \geq 1$ , then

$$\omega_{z,y}(s_J s_K^*) = \overline{z_{J_1}} y_{K_1} \omega_{z,y}(I) \sum_{|A|=\alpha m-i} \overline{(z^{\otimes \alpha})_{J_2 A}} (y^{\otimes \beta})_{K_2 A}. \quad (2.9)$$

(c) If  $(i, i') \neq (0, 0)$  and  $\alpha m - i \neq \beta l - i'$  for any  $\alpha, \beta \geq 1$ , then  $\omega_{z,y}(s_J s_K^*) \in \mathcal{W}_{z,y} := \text{Lin}\langle \{\omega_{z,y}(s_M), \omega_{z,y}(s_L^*) : 1 \leq |M| \leq m-1, 1 \leq |L| \leq l-1\} \rangle$ .

Here  $z_\emptyset = y_\emptyset := 1, z_{J_1} := z_{J_1^{(1)}} \cdots z_{J_1^{(a)}}$  and  $y_{K_1} := y_{K_1^{(1)}} \cdots y_{K_1^{(b)}}$  when  $J_1 = J_1^{(1)} \cdots J_1^{(a)}, K_1 = K_1^{(1)} \cdots K_1^{(b)}, |J_1^{(j)}| = m$  and  $|K_1^{(j')}| = l$  for  $j = 1, \dots, a$  and  $j' = 1, \dots, b$ .

- (ii) For  $\mathcal{W}_{z,y}$  in (i)(c),  $\mathcal{W}_{z,y} \subset \mathcal{X} := \text{Lin}\langle \{\omega_{z,y}(s_M) : 0 \leq |M| \leq m-1\} \rangle$ .

*Proof.* (i) From (2.7) and (2.8),  $\omega_{z,y}(X) = \omega_{z,y}(s(z)^* X s(y))$  for any  $X \in \mathcal{O}_n$ . From this,

$$\omega_{z,y}(s_J s_K^*) = \overline{z_{J_1}} y_{K_1} \omega_{z,y}(s_{J_2} s_{K_2}^*). \quad (2.10)$$

If  $(i, i') = (0, 0)$ , then (a) holds from (2.10). Assume  $(i, i') \neq (0, 0)$ . From (2.7) and (2.8),

$$\begin{aligned} \omega_{z,y}(s_{J_2} s_{K_2}^*) &= \omega_{z,y}((s(z)^*)^\alpha s_{J_2} s_{K_2}^* s(y)^\beta) \\ &= \sum_{|A|=\alpha m-i} \sum_{|B|=\beta l-i'} \overline{(z^{\otimes \alpha})_{J_2 A}} (y^{\otimes \beta})_{K_2 B} \omega_{z,y}(s_A^* s_B) \end{aligned} \quad (2.11)$$

for any  $\alpha, \beta \geq 1$ . From this, if  $\alpha m - i = \beta l - i'$ , then  $s_A^* s_B = \delta_{AB} I$  in (2.11). Hence (b) holds. If  $\alpha m - i \neq \beta l - i'$  for any  $\alpha, \beta \geq 1$ , then

$$\omega_{z,y}(s_{J_2} s_{K_2}^*) = \begin{cases} \sum_{|L|=l-i'} y_{K_2 L} \omega_{z,y}(s_{J_2} s_L) & \text{when } m-i > l-i', \\ \sum_{|L|=m-i} \overline{z_{J_2 L}} \omega_{z,y}(s_L^* s_{K_2}^*) & \text{when } m-i < l-i'. \end{cases} \quad (2.12)$$

If  $m-i > l-i'$ , then  $1 \leq |J_2| + |L| = i + l - i' < m$ . If  $m-i < l-i'$ , then  $1 \leq |K_2| + |L| = i' + m - i < l$ . From these and (2.10), (c) holds.

(ii) Assume  $1 \leq |K| \leq l-1$ . Then  $l - |K| = \gamma m + j$  for some  $\gamma \geq 0$  and  $0 \leq j \leq m-1$ . From (2.7),

$$\omega_{z,y}(s_K^*) = \omega_{z,y}((s(z)^*)^\gamma s_K^* s(y)) = \sum_{|J_1|=\gamma m} \sum_{|J_2|=j} \overline{(z^{\otimes \gamma})_{J_1}} y_{K J_1 J_2} \omega_{z,y}(s_{J_2}). \quad (2.13)$$

Hence  $\omega_{z,y}(s_K^*) \in \mathcal{X}$  and  $\mathcal{W}_{z,y} \subset \mathcal{X}$ .  $\blacksquare$

**Lemma 2.8** *Assume Assumption E for  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$  with  $\omega_{z,y}$  in (2.8).*

- (i) *Assume that  $d, \alpha, \beta \geq 1$  satisfy  $d = \alpha m = \beta l$  and  $d \geq 2$ . If  $m > l$  and  $z$  is nonperiodic, then  $\omega_{z,y}(s_J) = 0$  for any  $1 \leq |J| \leq d-1$ .*
- (ii) *If  $m = l \geq 2$  and  $z \not\sim y$ , then  $\omega_{z,y}(s_J) = 0$  for any  $1 \leq |J| \leq m-1$ .*
- (iii) *Let  $\omega$  be a sub-Cuntz state by  $z$  and  $m \geq 2$ . If  $z$  is nonperiodic, then  $\omega(s_J) = 0$  for any  $1 \leq |J| \leq m-1$ .*

*Proof.* Assume that  $d, \alpha, \beta \geq 1$  satisfy  $d = \alpha m = \beta l$  and  $d \geq 2$ . For  $1 \leq a \leq d-1$ , let  $J \in \{1, \dots, n\}^a$ . From (2.7),

$$\omega_{z,y}(s_J) = \omega_{z,y}((s(z)^\alpha)^* s_J s(y)^\beta) = \sum_{|K|=d-a} \sum_{|L|=a} \overline{(z^{\otimes \alpha})_{JK}} (y^{\otimes \beta})_{KL} \omega_{z,y}(s_L). \quad (2.14)$$

By rewriting this,

$$u_a = T_{d-a}(\overline{z^{\otimes \alpha}}) T_a(y^{\otimes \beta}) u_a \quad (2.15)$$

where  $T_a(z)$  is as in (2.1) and  $u_a := \sum_{|J|=a} \omega_{z,y}(s_J) e_J \in \mathcal{V}_{n,a}$ .



If  $\omega_{z,y}(s_J) \neq 0$  for some  $J \in \{1, \dots, n\}^a$  and  $1 \leq a \leq d-1$ , then  $u_a \neq 0$  in (2.15). Applying Lemma 2.6(i) to (2.15) with  $(m, X, Y, c, v) = (d, z^{\otimes \alpha}, y^{\otimes \beta}, 1, u_a)$ , we obtain

$$z^{\otimes \alpha} = z_1 \otimes z_2, \quad y^{\otimes \beta} = z_2 \otimes z_1 \quad (2.16)$$

for some  $z_1 \in (\mathcal{V}_{n,a})_1$  and  $z_2 \in (\mathcal{V}_{n,d-a})_1$ .

(i) If  $\omega_{z,y}(s_J) \neq 0$  for some  $J \in \{1, \dots, n\}^a$  and  $1 \leq a \leq d-1$ , we obtain (2.16). From Corollary A.6(iii) and  $m > l$ ,  $z$  is periodic. This contradicts with the assumption of  $z$ . Hence  $\omega_{z,y}(s_J) = 0$  for any  $1 \leq |J| \leq d-1$ .

(ii) Assume  $m = l$ . Then  $\alpha = \beta = 1$  and  $d = m$  in (2.15). If  $\omega_{z,y}(s_J) \neq 0$  for some  $J \in \{1, \dots, n\}^a$  and  $1 \leq a \leq m-1$ , then  $z = z_1 \otimes z_2$  and  $y = z_2 \otimes z_1$  for some  $z_1 \in (\mathcal{V}_{n,a})_1$  and  $z_2 \in (\mathcal{V}_{n,m-a})_1$  from (2.16) with  $\alpha = \beta = 1$ . Hence  $z \sim y$ . This contradicts with the assumption of  $z$  and  $y$ . Hence  $\omega_{z,y}(s_J) = 0$  for any  $1 \leq |J| \leq m-1$ .

(iii) Remark that Assumption EC holds for  $z$  and  $z$ . In this case,  $\omega = \omega_{z,z}$ . Hence (2.15) holds for  $z = y$ ,  $\alpha = \beta = 1$  and  $d = m$ . If  $\omega(s_J) \neq 0$ , then  $z_1 \otimes z_2 = z = z_2 \otimes z_1$  for some  $z_1 \in (\mathcal{V}_{n,a})_1$  and  $z_2 \in (\mathcal{V}_{n,m-a})_1$  from (2.16) with  $y = z$ . From Corollary A.6(i) with  $c = 1$ ,  $z$  is periodic. This contradicts with the assumption of  $z$ . Hence  $\omega(s_J) = 0$  for any  $1 \leq |J| \leq m-1$ . ■

### 3 Proofs of main theorems

#### 3.1 Proof of Theorem 1.4

When  $m = 1$ ,  $\omega$  is a Cuntz state. Hence it suffices to show the case of  $m \geq 2$ . For  $J, K \in \bigcup_{a \geq 0} \{1, \dots, n\}^a$ , we compute the value  $\omega(s_J s_K^*)$  as follows.

Assume  $|J| - |K| \in m\mathbb{Z}$ . Then either  $|J|, |K| \in m\mathbb{Z}_{\geq 0}$  or  $|J|, |K| \notin m\mathbb{Z}_{\geq 0}$  holds. If  $|J|, |K| \in m\mathbb{Z}_{\geq 0}$ , then  $\omega(s_J s_K^*) = \overline{z_J} z_K$  from Theorem 2.3(ii) where we use the notation in Theorem 1.4(iii). If  $|J|, |K| \notin m\mathbb{Z}_{\geq 0}$ , then the condition for  $J, K$  in “otherwise” of Theorem 1.4(iii) holds. In this case,

$$\begin{aligned} \omega(s_J s_K^*) &= \overline{z_{J_1}} z_{K_1} \omega(s_{J_2} s_{K_2}^*) \\ &= \overline{z_{J_1}} z_{K_1} \sum_{|L|=m-|J_2|} \omega(s_{J_2} s_L s_L^* s_{K_2}^*) \\ &= \overline{z_{J_1}} z_{K_1} \sum_{|L|=m-|J_2|} \overline{z_{J_2 L}} z_{K_2 L}. \end{aligned} \quad (3.1)$$

Assume  $|J| - |K| \notin m\mathbb{Z}$ . In Lemma 2.7, let  $l = m$  and  $y = z$ . Then the GNS representation  $(\mathcal{H}, \pi, \Omega)$  by  $\omega$  satisfies Assumption EC for  $\Omega_z = \Omega_y = \Omega$  with  $\omega = \omega_{z,y}$ . Applying Lemma 2.7(i)(c) to  $\omega$ ,

$$\omega(s_J s_K^*) \in \mathcal{W}_{z,z} = \text{Lin}\langle \{\omega(s_J), \omega(s_K^*) : 1 \leq |J|, |K| \leq m-1\} \rangle. \quad (3.2)$$

(i) Assume that  $z$  is nonperiodic. Then  $\omega(s_J) = 0$  for all  $1 \leq |J| \leq m-1$  from Lemma 2.8(iii). From (3.2),  $\omega(s_J s_K^*) = 0$  for all  $J, K$  which satisfy  $|J| - |K| \notin m\mathbb{Z}$ . From this and the case of  $|J| - |K| \in m\mathbb{Z}$ ,  $\omega(s_J s_K^*)$  is determined by only  $z$  for all  $J, K$ . Hence  $\omega$  is unique.

Assume  $z = x^{\otimes p}$  for some  $p \geq 2$  and  $x \in (\mathcal{V}_{n,m'})_1$ . Let  $\zeta := e^{2\pi\sqrt{-1}/p} \in U(1)$ . For  $1 \leq j \leq p$ , assume that  $\omega_j$  is a sub-Cuntz state on  $\mathcal{O}_n$  by  $\zeta^j x$ . Then we see that  $\omega_j$  is a sub-Cuntz state by  $z$  for all  $j$ . On the other hand, if  $i \neq j$ , then  $\omega_i \neq \omega_j$ . Hence a sub-Cuntz state by  $z$  is not unique.

(ii) From (i) and Proposition 2.1(ii), the statement holds.

(iii) From the proof of (i), the statement holds. ■

### 3.2 Proof of Theorem 1.5

**Lemma 3.1** *Let  $x = \sum x_J e_J \in (\mathcal{V}_{n,m})_1$  and  $p \geq 2$ . Assume that  $\mathcal{O}_n$  acts on a Hilbert space  $\mathcal{H}$  with a cyclic unit vector  $\Omega$  which satisfies*

$$s(x)^p \Omega = \Omega \quad (3.3)$$

where  $s(x) := \sum x_J s_J \in \mathcal{O}_n$ . Then the following hold:

(i) *There exist a unit vector  $(\alpha_i) \in \mathbb{R}^p$  and an orthogonal set  $\{\Omega_i\}_{i=1}^p \subset \mathcal{H}$  such that*

$$\Omega = \sum_{i=1}^p \alpha_i \Omega_i, \quad s(x) \Omega_i = \zeta^{-i} \Omega_i, \quad \|\Omega_i\| = 0 \text{ or } 1 \quad (i = 1, \dots, p) \quad (3.4)$$

where  $\zeta := e^{2\pi\sqrt{-1}/p}$ .

(ii) *In (i), define  $\omega_i := \langle \Omega_i | (\cdot) \Omega_i \rangle$ . If  $\Omega_i \neq 0$ , then  $\omega_i(s(x)^k) = \zeta^{-ik}$  for any  $k \geq 1$ .*

(iii) *Fix  $i, j \in \{1, \dots, p\}$  and define  $x' := \zeta^i x$  and  $x'' := \zeta^j x$ . Assume*

$m \geq 2$ . If  $i \neq j$ , then for  $J, K \in \bigcup_{a=1}^{m-1} \{1, \dots, n\}^a$ ,

$$\langle \Omega_i | s_J s_K^* \Omega_j \rangle = \begin{cases} \sum_{|L|=m-|J|} \overline{x'_{JL}} \langle \Omega_i | s_L^* s_K^* \Omega_j \rangle & \text{when } |J| > |K|, \\ 0 & \text{when } |J| = |K|, \\ \sum_{|L|=m-|K|} x''_{KL} \langle \Omega_i | s_J s_L \Omega_j \rangle & \text{when } |J| < |K|. \end{cases} \quad (3.5)$$

(iv) If  $i \neq j$ , then  $\langle \Omega_i | s_J s_K^* \Omega_j \rangle = 0$  for any  $J, K$ .

*Proof.* (i) Define  $\mathcal{K} := \{w \in \mathcal{H} : s(x)^p w = w\}$ . Since  $\Omega \in \mathcal{K}$ ,  $\mathcal{K}$  is a non-zero closed subspace of  $\mathcal{H}$ . Then  $R := s(x)|_{\mathcal{K}}$  satisfies  $R^p = I_{\mathcal{K}}$ . From this, we obtain the spectral decomposition  $R = \sum_{j=1}^p \zeta^{-j} E_j$  where  $\{E_j\}$  is the orthogonal set of projections on  $\mathcal{K}$  such that  $E_1 + \dots + E_p = I_{\mathcal{K}}$ . For  $i = 1, \dots, p$ , define  $\alpha_i := \|E_i \Omega\|$ , and  $\Omega_i := \alpha_i^{-1} E_i \Omega$  when  $\alpha_i \neq 0$  and  $\Omega_i := 0$  otherwise. Then the statement holds.

(ii) From (3.4), the statement holds.

(iii) From (3.4),

$$s(x') \Omega_i = \Omega_i, \quad s(x'') \Omega_j = \Omega_j. \quad (3.6)$$

In Lemma 2.7, let  $l = m$ ,  $z := x'$  and  $y := x''$ . Assumption E holds for  $z$  and  $y$  with  $\omega_{z,y} = \langle \Omega_i | (\cdot) \Omega_j \rangle$ . Assume  $|J| = |K| = a$ . From (i),  $\omega_{z,y}(I) = \langle \Omega_i | \Omega_j \rangle = 0$ . From this, the statement holds. The rest is proved from Lemma 2.7(i).

(iv) If  $m = 1$ , then  $\langle \Omega_i | s_J s_K^* \Omega_j \rangle = \overline{x'_J} x''_K \langle \Omega_i | \Omega_j \rangle = 0$  for any  $J, K$  when  $i \neq j$ . Hence we assume  $m \geq 2$ . From (iii) and Lemma 2.7(ii), it is sufficient to show  $\langle \Omega_i | s_J \Omega_j \rangle = 0$  for all  $1 \leq |J| \leq m-1$ . In Lemma 2.8(ii), let  $z := x'$  and  $y := x''$ . Since  $z \not\sim y$ ,  $\langle \Omega_i | s_J \Omega_j \rangle = 0$  for  $1 \leq |J| \leq m-1$  from Lemma 2.8(ii).  $\blacksquare$

*Proof of Theorem 1.5.* Assume that  $\omega$  is a sub-Cuntz state by  $z$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$ . For  $X \in \mathcal{O}_n$ , we write  $X$  as  $\pi(X)$  for the simplicity of description. Then the assumption in Lemma 3.1 is satisfied. For  $\alpha_i$  in Lemma 3.1(i), define  $a_i := \alpha_i^2$  for  $i = 1, \dots, p$ . From Lemma 3.1(i) and (iv),  $\omega = \sum_{i=1}^p |\alpha_i|^2 \omega_i = \sum_{i=1}^p a_i \omega_i$ . If  $\Omega_j \neq 0$ , then we see that  $\omega_j$  is the sub-Cuntz state by  $\zeta^j x$ . If  $\Omega_j = 0$ , then  $\omega_j = 0$  and  $a_j = 0$ . In this case,

we can replace  $\omega_j$  with the sub-Cuntz state by  $\zeta^j x$  with keeping  $\sum_{i=1}^p a_i \omega_i$ . Therefore (1.4) holds as a convex-hull of states.

We prove the uniqueness as follows. Assume  $\omega = \sum_{j=1}^p b_j \omega_j$  for some  $(b_1, \dots, b_p) \in \Delta_{p-1}$ . From Lemma 3.1(ii),

$$0 = \sum_{j=1}^p (a_j - b_j) \omega_j(s(x)^k) = \sum_{j=1}^p (a_j - b_j) \zeta^{-jk} \quad \text{for all } k \geq 1. \quad (3.7)$$

This implies  $a_j - b_j = 0$  for all  $j$ . Hence  $(a_1, \dots, a_p)$  is unique.  $\blacksquare$

### 3.3 Proof of Theorem 1.7

Let  $\sim$  be as in Theorem 1.7. By Theorem 2.3, we can prove the following.

**Lemma 3.2** *For two nonperiodic parameters  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$ , the following are equivalent:*

- (i)  $\tilde{\omega}_z \sim \tilde{\omega}_y$ .
- (ii) Assumption EC in Lemma 2.7 holds for  $z$  and  $y$ .

**Lemma 3.3** *Let  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$ .*

- (i) *Assume Assumption EC for  $z$  and  $y$  with  $\omega_{z,y}$  in (2.8). If both  $z$  and  $y$  are nonperiodic, then  $\omega_{z,y} \neq 0$ .*
- (ii) *Assume Assumption E for  $z$  and  $y$ . If there exist integers  $\alpha, \beta \geq 1$  such that  $\alpha m = \beta l$  and  $z^{\otimes \alpha} \neq y^{\otimes \beta}$ , then  $\omega_{z,y}(I) = 0$ . Especially, if  $m = l$  and  $z \neq y$ , then  $\omega_{z,y}(I) = 0$ .*

*Proof.* (i) Let  $\mathcal{H}$  be as in Assumption E. If  $\omega_{z,y} \equiv 0$ , then  $0 = \omega_{z,y}(s_J) = \langle \Omega_z | s_J \Omega_y \rangle$  for all  $J$ . Since  $\langle \Omega_y | (\cdot) \Omega_y \rangle = \tilde{\omega}_y$  and  $\mathcal{H}$  is generated by  $\{s_J \Omega_y : J\}$  from Lemma 2.4(ii),  $\Omega_z = 0$ . This contradicts with  $\|\Omega_z\| = 1$  in Assumption EC. Hence  $\omega_{z,y} \neq 0$ .

(ii) By (2.7) and (2.8),

$$\omega_{z,y}(I) = \omega_{z,y}((s(z)^*)^\alpha s(y)^\beta) = \langle z^{\otimes \alpha} | y^{\otimes \beta} \rangle \omega_{z,y}(I). \quad (3.8)$$

By Lemma A.1,  $z^{\otimes \alpha} \neq y^{\otimes \beta}$  implies  $\langle z^{\otimes \alpha} | y^{\otimes \beta} \rangle \neq 1$ . Hence  $\omega_{z,y}(I) = 0$ .  $\blacksquare$

**Lemma 3.4** *Let  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$ . If  $l = m$  and both  $z$  and  $y$  are nonperiodic, then the following are equivalent:*

(i)  $\tilde{\omega}_z \sim \tilde{\omega}_y$ .

(ii)  $z \sim y$ .

*Proof.* When  $m = 1$ , both  $\tilde{\omega}_z$  and  $\tilde{\omega}_y$  are Cuntz states. Hence it is sufficient to show the case of  $m \geq 2$ .

(i) $\Rightarrow$ (ii) Assume  $\tilde{\omega}_z \sim \tilde{\omega}_y$ . If  $z = y$ , then the statement holds. Assume  $z \neq y$ . From Lemma 3.2, Assumption EC for  $z$  and  $y$  holds. Let  $\omega_{z,y}$  be as in (2.8). Then  $\omega_{z,y}(I) = 0$  from Lemma 3.3(ii). From Lemma 3.3(i) and Lemma 2.7(ii), there must exist  $1 \leq a \leq m - 1$  such that  $\omega_{z,y}(s_J) \neq 0$  for some  $J \in \{1, \dots, n\}^a$ . From Lemma 2.8(ii),  $z \sim y$ .

(ii) $\Rightarrow$ (i) Assume  $z \sim y$  and  $z \neq y$ . Then there exist  $x_1, x_2 \in \bigcup_{a \geq 1} (\mathcal{V}_{n,a})_1$  such that  $z = x_1 \otimes x_2$  and  $y = x_2 \otimes x_1$ . Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\tilde{\omega}_z$ . From Theorem 2.3(ii),  $\pi(s(z))\Omega = \Omega$ . Then  $\Omega' := \pi(s(x_2))\Omega \in \mathcal{H}$  is also a cyclic unit vector because  $\tilde{\omega}_z$  is pure, and we can verify  $\pi(s(y))\Omega' = \Omega'$ . Hence Assumption EC for  $z$  and  $y$  holds. From Lemma 3.2,  $\tilde{\omega}_z \sim \tilde{\omega}_y$ .  $\blacksquare$

*Proof of Theorem 1.7.* (i) $\Rightarrow$ (ii) Assume  $\tilde{\omega}_z \sim \tilde{\omega}_y$ . From Lemma 3.2, Assumption EC for  $z$  and  $y$  holds. Let  $\omega_{z,y}$  be as in (2.8).

Assume  $m > l$ . From Corollary A.6(ii), there exist no  $\alpha, \beta$  such that  $z^{\otimes \alpha} = y^{\otimes \beta}$ . From this and Lemma 3.3(ii),  $\omega_{z,y}(I) = 0$ . Since  $z$  is nonperiodic,  $\omega_{z,y}(s_J) = 0$  for any  $1 \leq |J| \leq d - 1$  from Lemma 2.8(i). From Lemma 2.7(i) and (ii),  $\omega_{z,y} \equiv 0$ . From Lemma 3.3(i), Assumption EC does not hold. From Lemma 3.2,  $\tilde{\omega}_z \not\sim \tilde{\omega}_y$ . Hence  $m \not\geq l$ . As the same token, we obtain  $m \not\leq l$ . Hence  $m = l$ . From Lemma 3.4,  $z \sim y$ .

(ii) $\Rightarrow$ (i) Assume  $z \sim y$ . Then  $m = l$ . From Lemma 3.4,  $\tilde{\omega}_z \sim \tilde{\omega}_y$ .  $\blacksquare$

### 3.4 Proofs of Proposition 1.11 and Proposition 1.12

*Proof of Proposition 1.11.* Let  $\omega$  and  $\omega'$  be sub-Cuntz states by  $z \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n',l})_1$ , respectively. Then  $z^{\otimes l} \in (\mathcal{V}_{n,ml})_1$  and  $y^{\otimes m} \in (\mathcal{V}_{n',ml})_1$ . For any  $J \in \{1, \dots, nn'\}^{ml}$ , we can verify  $(\omega \otimes_{\varphi} \omega')(s_J^{(nn')}) = \overline{(z^{\otimes l} \boxtimes y^{\otimes m})_J}$  where  $\boxtimes$  is as in (1.12). Hence  $\omega \otimes_{\varphi} \omega'$  is a sub-Cuntz state by  $z^{\otimes l} \boxtimes y^{\otimes m} \in (\mathcal{V}_{nn',ml})_1$ .  $\blacksquare$

In this proof, there is no assumption of nonperiodicity for  $z$  and  $y$ . Hence  $\omega, \omega'$  and  $\omega \otimes_{\varphi} \omega'$  are not always unique.

- Lemma 3.5** (i) For  $x, x' \in \mathcal{V}_{n,m}$  and  $y, y' \in \mathcal{V}_{n',m}$ ,  $\langle x \boxtimes y | x' \boxtimes y' \rangle = \langle x | x' \rangle \langle y | y' \rangle$ .
- (ii) Let  $x, x' \in (\mathcal{V}_{n,m})_1$  and  $y, z \in (\mathcal{V}_{n',m})_1$ . If  $x \boxtimes y = x' \boxtimes z$  or  $y \boxtimes x = z \boxtimes x'$ , then  $y = cz$  for some  $c \in U(1)$ .
- (iii) If  $x \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n',m})_1$  satisfy  $x \boxtimes y = w^{\otimes p}$  for some  $w \in (\mathcal{V}_{nn',m'})_1$  and  $p \geq 2$ , then there exist  $v_1 \in (\mathcal{V}_{n,m'})_1$  and  $v_2 \in (\mathcal{V}_{n',m'})_1$  such that  $x = v_1^{\otimes p}$  and  $y = v_2^{\otimes p}$ .

*Proof.* (i) By definition, the statement holds.

(ii) Assume  $x \boxtimes y = x' \boxtimes z$ . By assumption and (i),  $1 = \langle x \boxtimes y | x' \boxtimes z \rangle = \langle x | x' \rangle \langle y | z \rangle$ . By applying Lemma A.1 to this, the statement holds. As the same token, the rest is proved.

(iii) By assumption,  $m'p = m$ . For  $J \in \{1, \dots, nn'\}^{m'}$ ,  $(x \boxtimes y)_{J^p} = (w_J)^p$  where  $J^p = J \cdots J$  ( $p$ -times). Hence  $x_{J_1^p} y_{J_2^p} = (w_{J_1 \boxtimes J_2})^p$  for all  $J_1 \in \{1, \dots, n\}^{m'}$  and  $J_2 \in \{1, \dots, n'\}^{m'}$ . Then there exist  $p$ -th roots  $A_{J_1}$  and  $B_{J_2}$  of  $x_{J_1^p}$  and  $y_{J_2^p}$  such that  $A_{J_1} B_{J_2} = w_{J_1 \boxtimes J_2}$ . Define  $A := \sum_{J_1} A_{J_1} e_{J_1} \in \mathcal{V}_{n,m'}$  and  $B := \sum_{J_2} B_{J_2} e_{J_2} \in \mathcal{V}_{n',m'}$ . Then  $w = A \boxtimes B$ . By normalizing  $A$  and  $B$ , we obtain two unit vectors  $w_1, w_2$  such that  $w = w_1 \boxtimes w_2$ . From these,  $x \boxtimes y = w^{\otimes p} = (w_1 \boxtimes w_2)^{\otimes p} = w_1^{\otimes p} \boxtimes w_2^{\otimes p}$ . From (ii),  $x = cw_1^{\otimes p}$  and  $y = \bar{c}w_2^{\otimes p}$  for some  $c \in U(1)$ . From these, we can choose  $v_1$  and  $v_2$  as the statement. ■

*Proof of Proposition 1.12.* Let  $\alpha, \beta, d$  be as in (1.13).

(i) Assume  $z * y = w^{\otimes p}$  for some  $w \in \mathcal{V}_{nn',k}$  and  $p \geq 2$  where  $k := d/p$ . By definition,  $z^{\otimes \alpha} \boxtimes y^{\otimes \beta} = w^{\otimes p}$ . From Lemma 3.5(iii), we obtain  $v_1 \in (\mathcal{V}_{n,k})_1$  and  $v_2 \in (\mathcal{V}_{n',k})_1$  such that

$$z^{\otimes \alpha} = v_1^{\otimes p}, \quad y^{\otimes \beta} = v_2^{\otimes p}. \quad (3.9)$$

From Corollary A.6(ii),  $v_1 = c'' z^{\otimes d_1}$  for some  $d_1 \geq 1$  and  $c'' \in U(1)$ . From this and (3.9),  $z^{\otimes \alpha} = v_1^{\otimes p} = (c'' z^{\otimes d_1})^{\otimes p} = (c'')^p z^{\otimes d_1 p}$ . Hence  $\alpha = pd_1$ . As the same token,  $\beta = pd_2$  for some  $d_2 \geq 1$ . Therefore  $\alpha$  and  $\beta$  have a common divisor  $p \geq 2$ . This contradicts with the choice of  $\alpha$  and  $\beta$ . Therefore  $z * y$  is nonperiodic.

(ii) Remark that  $\tilde{\omega}_{z*y}$  is uniquely defined by (i) and Theorem 1.4(i). We see that  $\{\tilde{\omega}_z \otimes_{\varphi} \tilde{\omega}_y\}(s_J^{(nn')}) = \overline{(z * y)_J}$  for all  $J \in \{1, \dots, nn'\}^d$ . Hence  $\tilde{\omega}_z \otimes_{\varphi} \tilde{\omega}_y$  is a sub-Cuntz state by  $z * y$ . Since a sub-Cuntz state by  $z * y$  is unique, the statement holds. ■

## 4 Examples

In this section, we show examples so that a reader can easily check main theorems.

### 4.1 Sub-Cuntz states of order 2

In this subsection, we show sub-Cuntz states on  $\mathcal{O}_n$  of order 2 as simplest, nontrivial and essentially new examples of main theorems. For convenience, we rewrite main theorems in § 1.2 for the case of  $m = 2$  as follows.

**Theorem 4.1** *Let  $((\mathbb{C}^n)^{\otimes 2})_1 := \{z \in \mathbb{C}^n \otimes \mathbb{C}^n : \|z\| = 1\}$ . Fix  $z = \sum_{ij} z_{ij} e_i \otimes e_j \in ((\mathbb{C}^n)^{\otimes 2})_1$ . Let  $\omega$  be a sub-Cuntz state on  $\mathcal{O}_n$  by  $z$ , that is,  $\omega$  is a state on  $\mathcal{O}_n$  which satisfies*

$$\omega(s_i s_j) = \overline{z_{ij}} \quad \text{for all } i, j = 1, \dots, n. \quad (4.1)$$

*Then such  $\omega$  exists for any  $z$  and the following hold:*

- (i)  *$\omega$  is unique if and only if  $z$  is nonperiodic, that is  $z \notin \{x \otimes x : x \in \mathbb{C}^n\}$ . In this case, we write  $\tilde{\omega}_z$  as  $\omega$ .*
- (ii) *If  $z$  is nonperiodic, then  $\tilde{\omega}_z$  is pure, and the following holds:*

$$\tilde{\omega}_z(s_J s_K^*) = \begin{cases} \overline{z_J} z_K & \text{when both } |J| \text{ and } |K| \text{ are even,} \\ \overline{z_{J_1}} z_{K_1} \sum_{d=1}^n \overline{z_{jd}} z_{kd} & \text{when } \begin{matrix} J = J_1 j, K = K_1 k, \\ \text{both } |J_1| \text{ and } |K_1| \text{ are even,} \end{matrix} \\ 0 & \text{when } |J| - |K| \text{ is odd} \end{cases} \quad (4.2)$$

*for  $J, K \in \bigcup_{a \geq 1} \{1, \dots, n\}^a \cup \{\emptyset\}$  where  $z_J := z_{J(1)} \cdots z_{J(l)}$  when  $J = J^{(1)} \cdots J^{(l)}$  and  $|J^{(i)}| = 2$  for  $i = 1, \dots, l$ .*

- (iii) *If  $z = x \otimes x$  for some  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$ , then there exists a real number  $0 \leq a \leq 1$  such that  $\omega$  has the following form*

$$\omega = a\omega_+ + (1-a)\omega_- \quad (4.3)$$

*where  $\omega_{\pm}$  denotes the Cuntz state on  $\mathcal{O}_n$  by  $\pm x$ , that is,  $\omega_{\pm}$  satisfies  $\omega_{\pm}(s_i) = \pm \overline{x_i}$  for all  $i$ .*

- (iv) Let  $z, y \in ((\mathbb{C}^n)^{\otimes 2})_1$ . If both  $z$  and  $y$  are nonperiodic, then  $\tilde{\omega}_z \sim \tilde{\omega}_y$  if and only if (a)  $z = y$ , or (b)  $z = x_1 \otimes x_2$  and  $y = x_2 \otimes x_1$  for some  $x_1, x_2 \in \mathbb{C}^n$ .

*Proof.* The existence of  $\omega$  holds from Fact 1.3.

- (i) From Theorem 1.4(i), the statement holds.
- (ii) From Theorem 1.4(ii) and (iii), statements hold.
- (iii) From the case of  $(p, m') = (2, 1)$  in Theorem 1.5, the statement holds.
- (iv) From Theorem 1.7, the statement holds. ■

We show a more convenient corollary as follows.

**Corollary 4.2** *Assume the same assumption in Theorem 4.1 for  $z = \sum_{ij} z_{ij} e_i \otimes e_j$ .*

- (i) *If  $A := (z_{ij}) \in M_n(\mathbb{C})$  satisfies  $\|A\| < 1$ , then  $\omega$  is unique and pure.*
- (ii) *If  $z_{ij} \neq z_{ji}$  for some  $i, j$ , then  $\omega$  is unique and pure.*

*Proof.* (i) Remark that  $A$  coincides with  $T_1(z)$  in (2.1) as operators on  $\mathbb{C}^n$ . The assumption implies that  $z$  is indecomposable. Especially,  $z$  is nonperiodic. From Theorem 4.1(i) and (ii), the statement holds.

(ii) In this case,  $z$  is nonperiodic. Hence the statement holds from Theorem 4.1(i) and (ii). ■

Next, we show concrete examples. In stead of  $z = \sum_{ij} z_{ij} e_i \otimes e_j \in ((\mathbb{C}^n)^{\otimes 2})_1$ , we use a matrix  $A = (z_{ij}) \in M_n(\mathbb{C})$  such that  $\|A\|_2 = 1$  in order to apply Corollary 4.2. We assume that  $\mathcal{O}_n$  acts on a Hilbert space with a cyclic unit vector  $\Omega$ . Define the vector state  $\omega$  on  $\mathcal{O}_n$  with respect to  $\Omega$ :

$$\omega := \langle \Omega | (\cdot) \Omega \rangle. \quad (4.4)$$

**Example 4.3** Let  $(c_i) \in \mathbb{C}^n$  be a unit vector. Assume that the following equation holds:

$$\sum_{i=1}^n c_i s_i^2 \Omega = \Omega. \quad (4.5)$$

- (i) If  $|c_i| < 1$  for all  $i$ , then  $A := \text{diag}(c_1, \dots, c_n) \in M_n(\mathbb{C})$  satisfies  $\|A\| < 1$ . Hence  $\omega$  is unique and pure from Corollary 4.2(i).



- (ii) If there exists  $i$  such that  $|c_i| = 1$ , then  $c_i s_i^2 \Omega = \Omega$ . Let  $q \in U(1)$  be a quadratic root of  $c_i$ . From Theorem 4.1(iii), there exists  $0 \leq a \leq 1$  such that  $\omega = a\omega_+ + (1-a)\omega_-$  where  $\omega_\pm$  denotes the Cuntz state by  $\pm q e_i$ , that is,  $\omega_\pm$  satisfies  $\omega_\pm(s_i) = \pm \bar{q}$ . In this case,  $\omega$  is pure if and only if  $a = 0$  or  $1$ .

Fix  $n = 2$  from here. Let  $s_{ij} := s_i s_j$  for  $i, j = 1, 2$ .

**Example 4.4** Assume that the following equation holds:

$$\frac{1}{2}(s_{11} - s_{12} + s_{21} + s_{22})\Omega = \Omega. \quad (4.6)$$

Then  $(z_{ij}) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$  satisfies  $z_{12} \neq z_{21}$ . From Corollary 4.2(ii),  $\omega$  is unique and pure.

**Example 4.5** Assume that the following equation holds:

$$\frac{1}{\sqrt{2}}(s_{12} + s_{21})\Omega = \Omega. \quad (4.7)$$

Then  $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  satisfies  $\|A\| = \frac{1}{\sqrt{2}} < 1$ . From Corollary 4.2(i),  $\omega$  is unique and pure.

## 4.2 Sub-Cuntz states associated with permutative representations

In this subsection, we show known results in § 5 of [7, 33] by using results of sub-Cuntz states. A representation  $(\mathcal{H}, \pi)$  of  $\mathcal{O}_n$  is said to be *permutative* if there exists an orthonormal basis  $B = \{v_k : k \in \Lambda\}$  of  $\mathcal{H}$  such that  $\pi(s_i)v_k \in B$  for any  $i, k$  [7, 13, 14]. We explain sub-Cuntz states associated with permutative representations as follows. For  $z = \sum z_J e_J \in (\mathcal{V}_{n,m})_1$ , assume  $z_J = 1$  for some  $J$ . In this case,  $z = e_J$  and the following holds.

**Proposition 4.6** (i) *For any  $J \in \{1, \dots, n\}^m$ , there exists a state  $\omega$  on  $\mathcal{O}_n$  which satisfies*

$$\omega(s_J) = 1. \quad (4.8)$$

- (ii) *If a state  $\omega$  on  $\mathcal{O}_n$  satisfies (4.8), then  $\omega$  is unique if and only if  $J$  is nonperiodic, that is,  $J = K^p$  for some  $K$  implies  $p = 1$ . In this case,  $\omega$  is a pure sub-Cuntz state  $\tilde{\omega}_{e_J}$  by  $e_J$ , and we write  $\phi_J$  as  $\tilde{\omega}_{e_J}$  for short.*

- (iii) If  $J$  is nonperiodic, then the GNS representation by  $\phi_J$  is permutative.
- (iv) If  $J = K^p$  for some nonperiodic word  $K$  and  $p \geq 2$ , then  $\phi_J = \sum_{j=1}^p a_j \omega_j$  for some nonnegative numbers  $a_1, \dots, a_p$  such that  $a_1 + \dots + a_p = 1$  where  $\omega_j$  denotes the sub-Cuntz state by  $e^{2\pi j\sqrt{-1}/p} e_J$ .
- (v) For two nonperiodic words  $J$  and  $K$ ,  $\phi_J \sim \phi_K$  if and only if  $J$  and  $K$  are conjugate, that is,  $J = K$  or  $J = L_1 L_2$  and  $K = L_2 L_1$  for some  $L_1, L_2$ .
- (vi) Let  $\mathfrak{S}_n$  denote the symmetric group on the set  $\{1, \dots, n\}$ . Define the action of  $\mathfrak{S}_n$  on  $\mathbb{C}^n$  as  $\sigma e_i := e_{\sigma(i)}$  for  $i = 1, \dots, n$   $\sigma \in \mathfrak{S}_n$ . With respect to this action, we identify  $\mathfrak{S}_n$  with the subgroup of  $U(n)$ . Then for any nonperiodic word  $J$ ,  $\alpha_\sigma^* \circ \phi_J = \phi_{\sigma J}$  for any  $\sigma \in \mathfrak{S}_n$  where  $\alpha^*$  is as in § 1.3.1 and  $\sigma J := (\sigma(j_1), \dots, \sigma(j_l))$  when  $J = (j_1, \dots, j_l)$ .
- (vii) Let  $\otimes_\varphi$  and  $\boxtimes$  be as in § 1.3.2. For two nonperiodic words  $J$  and  $K$ ,  $\phi_J \otimes_\varphi \phi_K = \phi_{J * K}$  where  $J * K := J^\alpha \boxtimes K^\beta$  such that  $\alpha, \beta \geq 1$  and  $\alpha|J| = \beta|K|$  is the least common multiple of  $|J|$  and  $|K|$ .

*Proof.* (i) Let  $z := e_J \in (\mathcal{V}_{n,m})_1$  when  $|J| = m$ . Then we see that  $\omega$  satisfies  $\omega(s_K) = \delta_{JK}$  for all  $K \in \{1, \dots, n\}^m$ . Hence  $\omega$  is a sub-Cuntz state by  $z$ . From Fact 1.3, the statement holds.

(ii) Let  $z := e_J \in (\mathcal{V}_{n,m})_1$  when  $|J| = m$ . By assumption,  $z$  is nonperiodic if and only if  $J$  is nonperiodic. From Theorem 1.4(i) and (ii), the statement holds.

(iii) Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\phi_J$ . We prove that  $(\mathcal{H}, \pi)$  is permutative as follows. Since  $\pi(s_J)\Omega = \Omega$ ,  $\phi_J(s_K) = 1$  when  $K \in X := \{J^a : a \geq 0\}$  and  $\phi_J(s_K) = 0$  when  $K \notin X$ . Let  $v_K := \pi(s_K)\Omega$ . Since  $J$  is nonperiodic,  $\langle v_K | v_L \rangle = 1$  when  $K = LJ^a$  or  $L = KJ^a$  for some  $a \geq 0$  and  $\langle v_K | v_L \rangle = 0$  otherwise. From Lemma A.1,  $\langle v_K | v_L \rangle = 1$  if and only if  $v_K = v_L$ . Therefore  $\{u \in \mathcal{H} : \text{there exists } K \text{ such that } u = v_K\}$  is an orthonormal basis of  $\mathcal{H}$  from Lemma 2.4(ii). Hence  $(\mathcal{H}, \pi)$  is permutative.

(iv) From Theorem 1.5, the statement holds.

(v) From Theorem 1.7, the statement holds.

(vi) From Proposition 1.10, the statement holds.

(vii) From Proposition 1.12, the statement holds. ■

When  $m = 1$  in Proposition 4.6, there exists  $i \in \{1, \dots, n\}$  such that  $z = e_i$ . In this case,  $z$  is nonperiodic and the following holds:

$$\omega(s_j) = \delta_{ij} \quad \text{for all } j = 1, \dots, n. \quad (4.9)$$

Any Cuntz state is given as a transformation of this by the dual action of the standard  $U(n)$ -action on  $\mathcal{O}_n$  (see the proof of Theorem B.1(ii)).

**Fact 4.7** *Assume that  $\mathcal{O}_n$  acts on the Hilbert space  $\ell^2(\Lambda)$  with an orthonormal basis  $B = \{v_\lambda : \lambda \in \Lambda\}$  such that  $s_i v_\lambda \in B$  for any  $i$  and  $\lambda$ , and  $\omega$  is the vector state on  $\mathcal{O}_n$  by  $v_{\lambda_0}$  for some  $\lambda_0 \in \Lambda$ . If  $\omega$  is finitely correlated, then  $\omega_L := \omega(s_L(\cdot)s_L^*)$  is a sub-Cuntz state for some  $L \in \{1, \dots, n\}^k$ . Especially, if  $\omega$  is pure, then  $\omega_L$  is also pure and  $\omega_L \sim \omega$ .*

*Proof.* Let  $\Omega := v_{\lambda_0}$ . By assumption, the action of  $\mathcal{O}_n$  on  $\mathcal{H}$  is permutative. Hence  $s_J^* \Omega = 0$  or  $s_J^* \Omega \in B$  for any  $J$ . Therefore  $\{s_J^* \Omega : J\} \subset B \cup \{0\}$  and  $\#\{s_J^* \Omega : J\} = \#(\{s_J^* \Omega : J\} \cap B) + 1$ . By assumption,

$$\infty > \dim \text{Lin}\langle \{s_J^* \Omega : J\} \rangle = \dim \text{Lin}\langle \{s_J^* \Omega \in B : J\} \rangle = \#\{s_J^* \Omega \in B : J\}. \quad (4.10)$$

Hence  $\#\{s_J^* \Omega : J\} < \infty$ . For any  $\lambda \in \Lambda$ , there exists a unique  $i$  such that  $s_i^* v_\lambda \in B$ . Hence there exists a unique sequence  $\{J^{(l)} \in \bigcup_{a \geq 1} \{1, \dots, n\}^a : |J^{(l)}| = l \text{ for all } l\}$  such that  $s_{J^{(l)}}^* \Omega \neq 0$  for any  $l$ . Since  $\#\{s_J^* \Omega : J\} < \infty$ , there exist  $p \geq 1$  and  $l_0 \geq 1$  such that  $s_{J^{(l_0+p)}}^* \Omega = s_{J^{(l_0)}}^* \Omega$ . Let  $L := J^{(l_0)}$  and  $\Omega' := s_L^* \Omega$ . Then  $s_{J'}^* \Omega' = \Omega'$  for some  $J'$ . This implies that  $\omega_L$  is a sub-Cuntz state by  $z = e_{J'}$ . If  $\omega$  is pure, then the statement holds by the construction of  $\omega_L$ .  $\blacksquare$

### 4.3 Infinitely correlated states as non-sub-Cuntz states

From Lemma 2.4(i), any infinitely correlated state is not a sub-Cuntz state on  $\mathcal{O}_n$  when  $n < \infty$ . In this subsection, we show examples of infinitely correlated states.

**Example 4.8** *(Infinitely correlated state associated with a permutative representation)* Let  $\mathbb{N} := \{1, 2, \dots\}$  and let  $\{e_{k,m} : (k,m) \in \mathbb{N} \times \mathbb{Z}\}$  denote the standard basis of  $\ell^2(\mathbb{N} \times \mathbb{Z})$ . For  $2 \leq n < \infty$ , define a representation  $\pi$  of  $\mathcal{O}_n$  on  $\ell^2(\mathbb{N} \times \mathbb{Z})$  by

$$\pi(s_i)e_{k,m} := e_{n(k-1)+i,m+1} \quad ((k,m) \in \mathbb{N} \times \mathbb{Z}, i = 1, \dots, n). \quad (4.11)$$

By definition,  $(\ell^2(\mathbb{N} \times \mathbb{Z}), \pi)$  is permutative, and  $\pi(s_1^m)^* e_{1,0} = e_{1,-m}$  for any  $m \geq 1$ . Hence  $\dim \text{Lin}\langle \{\pi(s_J)^* e_{1,0} : J\} \rangle = \infty$ . Therefore the state  $\omega := \langle e_{1,0} | \pi(\cdot) e_{1,0} \rangle$  is infinitely correlated.

**Example 4.9** (Quasi-free states) We show that any quasi-free state on  $\mathcal{O}_n$  is infinitely correlated. Let  $\Lambda_n := \{(a_1, \dots, a_n) \in \mathbb{R}^n : a_i > 0 \text{ for all } i, a_1 + \dots + a_n = 1\}$ . For  $a \in \Lambda_n$ , define the state  $\rho_a$  on  $\mathcal{O}_n$  by

$$\rho_a(s_J s_K^*) := a_J \delta_{JK} \quad (J, K \in \mathcal{I} := \bigcup_{m \geq 0} \{1, \dots, n\}^m) \quad (4.12)$$

where  $a_J := a_{j_1} \cdots a_{j_m}$  for  $J = (j_1, \dots, j_m)$  and  $a_\emptyset := 1$ . The state  $\rho_a$  is called the *quasi-free state* on  $\mathcal{O}_n$  by  $a$  [3, 19]. It is known that the GNS representation by  $\rho_a$  is a type III factor representation;  $\rho_a \sim \rho_b$  if and only if  $a = b$ ;  $\rho_a \otimes_\varphi \rho_b = \rho_{a \boxtimes b}$  [25, 36, 39].

Fix  $a \in \Lambda_n$  and let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\rho_a$ . For  $J \in \mathcal{I}$ , define  $v_J := a_J^{-1/2} \pi(s_J)^* \Omega \in \mathcal{H}$ . From (4.12), we see that  $\{v_J : J \in \mathcal{I}\}$  is an orthonormal system in  $\mathcal{H}$ . Therefore  $\dim \text{Lin}\langle \{\pi(s_J)^* \Omega : J \in \mathcal{I}\} \rangle = \dim \text{Lin}\langle \{v_J : J \in \mathcal{I}\} \rangle = \#\mathcal{I} = \infty$ . Hence  $\rho_a$  is infinitely correlated.

## Appendix

### A Combinatorics on words in tensor algebra

In this section, we prove the freeness of some semigroup with uncountable rank associated with a tensor algebra. By using this fact and known results about free semigroups, we derive crucial lemmas for main theorems.

#### A.1 Cancellation law and equidivisibility of tensor product

We show the cancellation law ([24], 2.6.1) and equidivisibility ([24], § 7.1) of tensor product. Let  $U(1) := \{c \in \mathbb{C} : |c| = 1\}$ . The following is very elementary, but mistakes are often found in literature.

**Lemma A.1** *If  $z$  and  $y$  are unit vectors in a Hilbert space, then  $\langle z|y \rangle = 1$  if and only if  $z = y$ .*

*Proof.* If  $\langle z|y \rangle = 1$ , then  $1 = |\langle z|y \rangle| \leq \|z\| \cdot \|y\| = 1$ . Since  $|\langle z|y \rangle| = \|z\| \cdot \|y\|$ ,  $z = cy$  for some  $c \in U(1)$ . From this,  $1 = \langle z|y \rangle = \langle cy|y \rangle = \bar{c}$ . Hence  $z = y$ . The inverse direction is trivial.  $\blacksquare$

**Lemma A.2** *Let  $(\mathcal{V}_{n,m})_1$  be as in §1.2.*

- (i) (*Cancellation law*) Assume that  $x, x' \in (\mathcal{V}_{n,m})_1$  and  $y, z \in \bigcup_{a \geq 1} (\mathcal{V}_{n,a})_1$  satisfy  $x \otimes y = x' \otimes z$  or  $y \otimes x = z \otimes x'$ . Then  $x = cx'$  and  $y = \bar{c}z$  for some  $c \in U(1)$ . In addition, if  $x = x'$ , then  $y = z$ .
- (ii) (*Equidivisibility*) Assume  $x \in (\mathcal{V}_{n,m})_1$ ,  $y \in (\mathcal{V}_{n,l})_1$  and  $m > l$ .
- (a) If  $x \otimes w = y \otimes z$ , then there exists  $x' \in (\mathcal{V}_{n,m-l})_1$  such that  $x = y \otimes x'$ .
- (b) If  $w \otimes x = z \otimes y$ , then there exists  $x'' \in (\mathcal{V}_{n,m-l})_1$  such that  $x = x'' \otimes y$ .

*Proof.* (i) By using Lemma A.1, the statement can be verified.

(ii) When  $x = \sum x_J e_J$  and  $y = \sum y_K e_K$ , define

$$x' := \sum_{|J_1|=l} \sum_{|J_2|=m-l} x_{J_1 J_2} \overline{y_{J_1}} e_{J_2} \in \mathcal{V}_{n,m-l}. \quad (\text{A.1})$$

Since  $x' = \sum_{|K|=m-l} \langle y \otimes e_K | z \rangle e_K$ ,

$$\|x'\|^2 = \sum_K |\langle y \otimes e_K | z \rangle|^2 \leq \sum_i |\langle v_i | z \rangle|^2 = \|z\|^2 = 1 \quad (\text{A.2})$$

where  $\{v_i\}$  is an orthonormal basis of  $\mathcal{V}_{n,m}$  such that  $\{y \otimes e_K : |K| = m-l\} \subset \{v_i\}$ . Then we can verify  $\langle x \otimes w | y \otimes z \rangle = \langle x' \otimes w | z \rangle$ . From this,

$$\begin{aligned} 1 &= \langle x \otimes w | y \otimes z \rangle \\ &= \langle x' \otimes w | z \rangle \\ &= \langle y \otimes x' \otimes w | y \otimes z \rangle \\ &= \langle y \otimes x' \otimes w | x \otimes w \rangle \\ &= \langle y \otimes x' | x \rangle. \end{aligned} \quad (\text{A.3})$$

From this and (A.2),  $1 = \langle y \otimes x' | x \rangle \leq \|y \otimes x'\| \cdot \|x\| \leq 1$ . This implies  $y \otimes x' = cx$  for some  $c \in \mathbb{C}$ . By substituting this into (A.3),  $1 = \langle y \otimes x' | x \rangle = \langle cx | x \rangle = \bar{c}$ . Hence  $y \otimes x' = x$  and  $\|x'\| = 1$ . Hence (a) is proved. As the same token, (b) is verified.  $\blacksquare$

## A.2 Projective homogeneous tensor semigroup is free

Let  $V := \mathcal{V}_{n,1}$  and we identify  $\mathcal{V}_{n,m}$  with  $V^{\otimes m}$  for  $m \geq 1$ , and let  $V^{\otimes 0} := \mathbb{C}$ . By forgetting the addition of the tensor algebra  $T(V) := \bigoplus_{m \geq 0} V^{\otimes m}$  over  $V$ ,  $T(V)$  is regarded as a semigroup with respect to the tensor product  $\otimes$ .

Furthermore, its projective space  $PT(V) := (T(V) \setminus \{0\})/\mathbb{C}^\times$  is a semigroup with respect to the product  $[x][y] := [x \otimes y]$  for  $x, y \in T(V) \setminus \{0\}$  where  $[x] := \{cx : c \in \mathbb{C}^\times\} \in PT(V)$ . For any subsemigroup  $S$  of  $T(V)$ ,  $PS := (S \setminus \{0\})/\mathbb{C}^\times$  is a subsemigroup of  $PT(V)$ . Especially, we consider the following subsemigroup  $G$  of  $(T(V), \otimes)$  and its projective semigroup  $PG$ :

$$G := \bigcup_{m \geq 1} (V^{\otimes m})_1, \quad (\text{A.4})$$

that is,  $G$  is the subsemigroup of all homogeneous unit vectors in  $T(V)$  except vectors in  $V^{\otimes 0}$ .

A semigroup  $S$  is said to be *free* if there exists a nonempty subset  $B$  of  $S$  such that  $B$  generates  $S$ , and for any semigroup  $S'$  and any map  $f$  from  $B$  to  $S'$ , there exists a homomorphism  $\hat{f}$  from  $S$  to  $S'$  such that  $\hat{f}|_B = f$  [24]. In this case,  $S$  is called the *free semigroup over  $B$*  and  $\#B$  is called the *rank* of  $S$ . A free semigroup is defined uniquely up to an isomorphism by the rank.

**Lemma A.3** ([11], Theorem 9.1) *A semigroup  $S$  is free if and only if there exists a nonempty subset  $B$  of  $S$  such that every element of  $S$  has a unique expression as a product of elements of  $B$ .*

**Proposition A.4** *For  $G$  in (A.4), its projective semigroup  $PG$  is free.*

*Proof.* Let  $\mathfrak{I}_n$  be as in Corollary 1.9. From Lemma A.3, it is sufficient to show that every element of  $PG$  can be expressed uniquely as a product of elements of  $B := P\mathfrak{I}_n$ .

Let  $X \in PG$ . By definition,  $X \in P(\mathcal{V}_{n,m})_1$  for some  $m \geq 1$ . Hence  $X = [x]$  for some  $x \in (\mathcal{V}_{n,m})_1$ . If  $x \in \mathfrak{I}_n$ , then  $X \in B$ . Hence  $X$  is uniquely written as an element of  $B$ . Assume  $x \notin \mathfrak{I}_n$ . By definition,  $x = z_1 \otimes z_2$  for some  $z_1, z_2 \in \bigcup_{a \geq 1} (\mathcal{V}_{n,a})_1$ . When  $y \in (\mathcal{V}_{n,m})_1$ , define  $|y| := m$ . Then  $1 \leq |z_1|, |z_2| < m = |x|$ . By decomposing  $x$  repeatedly, we can obtain a finest decomposition  $x = x_1 \otimes \cdots \otimes x_l$ . Then  $x_i \in \mathfrak{I}_n$  for  $i = 1, \dots, l$  and  $l \leq m$ . Hence  $X = [x]$  always has a decomposition  $[x_1] \cdots [x_l]$  for  $[x_i] \in B$  for all  $i$ . Assume that  $X = X'_1 \cdots X'_k$  and  $X'_j = [x'_j]$  for  $x'_j \in \mathfrak{I}_n$  for all  $j$ . Then  $x_1 \otimes \cdots \otimes x_l = cx'_1 \otimes \cdots \otimes x'_k$  for some  $c \in U(1)$ . From Lemma A.2(ii),  $|x_1| = |x'_1|$  because  $x_1, x'_1 \in \mathfrak{I}_n$ . From this and Lemma A.2(i),  $x_1 = c_1 x'_1$  for some  $c_1 \in U(1)$ . This implies  $X_1 = X'_1$ . By the mathematical induction, we can verify that  $X_i = X'_i$  for all  $i$  and  $l = k$ . Therefore  $X$  has a unique expression as a product of elements of  $B$ .  $\blacksquare$

Remark that  $T(V)$  is the free  $\mathbb{C}$ -algebra over the set  $\{1, \dots, n\}$  when  $\dim V = n$  [43]. On the other hand,  $PG$  is the free semigroup over the uncountable set  $P\mathfrak{J}_n$ , that is, the rank of  $PG$  is uncountable.

**Proposition A.5** *Let  $B^+$  denote the free semigroup over a set  $B$ .*

- (i) ([24], Proposition 7.1.5) *Let  $\#B \geq 2$ , and let  $u, v \in B^+$ . Then  $uv = vu$  if and only if  $u$  and  $v$  are powers of the same element  $w \in B^+$ .*
- (ii) ([24], Proposition 7.1.6) *If  $u, v \in B^+$  satisfy  $u^m = v^n$  for some  $m, n \geq 1$ , then  $u$  and  $v$  are powers of the same element  $w \in B^+$ .*
- (iii) ([47], Proposition 1.3.3) *Recall the definition of conjugacy in Theorem 1.7(ii). Let  $x, y \in B^n := \{b_1 \cdots b_n : b_i \in B, i = 1, \dots, n\}$  and  $s, t$  be nonperiodic such that  $x = s^p$  and  $y = t^q$ . Then  $x$  and  $y$  are conjugate if and only if  $s$  and  $t$  are conjugate.*

**Corollary A.6** *Let  $x \in (\mathcal{V}_{n,m})_1$  and  $y \in (\mathcal{V}_{n,l})_1$ .*

- (i) *Assume*

$$y \otimes x = cx \otimes y \tag{A.5}$$

*for some  $c \in U(1)$ . Then there exists  $w \in (\mathcal{V}_{n,a})_1$  such that  $x = \gamma_1 w^{\otimes f_1}$  and  $y = \gamma_2 w^{\otimes f_2}$  for some  $f_1, f_2 \geq 1$  and  $\gamma_1, \gamma_2 \in U(1)$ . Especially,  $x \otimes y$  is periodic and  $c = 1$ .*

- (ii) *Assume that there exist two integers  $\alpha, \beta \geq 1$  such that*

$$x^{\otimes \alpha} = y^{\otimes \beta}. \tag{A.6}$$

*Then there exists  $w$  such that  $x = \gamma_1 w^{\otimes k_1}$  and  $y = \gamma_2 w^{\otimes k_2}$  for some  $k_1, k_2 \geq 1$  and  $\gamma_1, \gamma_2 \in U(1)$ . Especially, if  $m > l$ , then  $x$  is periodic. If  $x$  is nonperiodic, then  $y = cx^{\otimes d}$  for some  $d \geq 1$  and  $c \in U(1)$ .*

- (iii) *Assume that  $m > l$  and there exist  $z_1, z_2$  such that*

$$z^{\otimes \alpha} = z_1 \otimes z_2, \quad y^{\otimes \beta} = z_2 \otimes z_1 \tag{A.7}$$

*for some  $\alpha, \beta$ . Then  $z$  is periodic.*

*Proof.* From Proposition A.4 and its proof, Proposition A.5 can be applied to the pair  $(B^+, B) = (PG, P\mathfrak{J}_n)$ .

- (i) From (A.5),  $[y][x] = [x][y]$  in  $PG$ . From Proposition A.5(i),  $[x], [y] \in \{W^p : p \geq 1\}$  for some  $W \in PG$ . Since  $W = [w]$  for some  $w \in G$ , we obtain the statement.

(ii) From (A.6),  $[x]^\alpha = [y]^\beta$  in  $PG$ . From Proposition A.5(ii), the statement holds.

(iii) Assume that  $z = u^{\otimes p}$  and  $y = v^{\otimes q}$  for some nonperiodic elements  $u$  and  $v$ . From Proposition A.5(iii),  $[u]$  and  $[v]$  are conjugate in  $PG$ . This implies  $u, v \in (\mathcal{V}_{n,k})_1$  for some  $k \geq 1$ . Hence  $z = u^{\otimes p} \in (\mathcal{V}_{n,kp})_1$  and  $y = v^{\otimes q} \in (\mathcal{V}_{n,kq})_1$ . Therefore  $m = kp$  and  $l = kq$ . Since  $m > l$ ,  $p > q \geq 1$ . Therefore  $z$  is periodic.  $\blacksquare$

## B Proofs of properties of Cuntz states

In this section, we prove well-known basic properties of Cuntz states on  $\mathcal{O}_n$  ( $2 \leq n \leq \infty$ ) [7]. Since both Fact 1.3 and Theorem 1.4(ii) are proved by using properties of Cuntz states, we do not use results of sub-Cuntz states here. Let  $\mathbb{N} := \{1, 2, 3, \dots\}$  and let  $s_1, s_2, \dots$  denote the Cuntz generators of  $\mathcal{O}_n$ . Define  $\mathcal{I} := \bigcup_{a \geq 0} \mathcal{I}_1^a$  where  $\mathcal{I}_1 := \{1, \dots, n\}$  when  $n < \infty$ , and  $\mathcal{I}_1 := \mathbb{N}$  when  $n = \infty$ . Define  $\mathfrak{h} := \ell^2(\mathcal{I}_1)$  and  $\mathfrak{h}_1 := \{z \in \mathfrak{h} : \|z\| = 1\}$ . Here we identify  $\mathfrak{h}$  with the set of all complex sequences  $(z_i)_{i \in \mathcal{I}_1}$  such that  $\sum_i |z_i|^2 < \infty$ .

**Theorem B.1** *Fix  $2 \leq n \leq \infty$ .*

- (i) *There exists a unique state  $\omega_1$  on  $\mathcal{O}_n$  such that  $\omega_1(s_1) = 1$ . In this case,  $\omega_1$  is pure and  $\omega_1(s_i) = 0$  when  $i \neq 1$ .*
- (ii) *For any  $z \in \mathfrak{h}_1$ , a Cuntz state on  $\mathcal{O}_n$  by  $z$  exists uniquely and is pure.*
- (iii) *For  $z \in \mathfrak{h}_1$ , let  $\omega_z$  denote the Cuntz state by  $z$ . Then  $\omega_z \sim \omega_y$  if and only if  $z = y$ .*
- (iv) *For any  $J, K$ ,  $\omega_z(s_J s_K^*) = \overline{z_J} z_K$ .*

*Proof.* (i) Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\omega_1$ . Then we see  $\pi(s_i)^* \Omega = \delta_{i1} \Omega$  for all  $i$ . This implies that  $\omega_1(s_J s_K^*) = 1$  when  $J, K \in W := \{\emptyset, (1), (11), (111), \dots\} \subset \mathcal{I}$ , and  $\omega_1(s_J s_K^*) = 0$  otherwise. Therefore the uniqueness of  $\omega_1$  holds.

We prove the existence and purity as follows. Let  $\{e_k : k \in \mathbb{N}\}$  denote the standard basis of  $\ell^2(\mathbb{N})$ .

Assume  $n < \infty$ . Define the action of  $\mathcal{O}_n$  on  $\ell^2(\mathbb{N})$  by

$$s_i e_k := e_{n(k-1)+i} \quad (i = 1, \dots, n, k \in \mathbb{N}). \quad (\text{B.1})$$



Since  $s_1 e_1 = e_1$ ,  $\omega_1 := \langle e_1 | (\cdot) e_1 \rangle$  satisfies  $\omega_1(s_1) = 1$ . Therefore the existence is proved. Next we prove the irreducibility of the action (B.1). Remark that any  $k \in \mathbb{N}$  is uniquely written as  $n(k' - 1) + i$  for some  $i = 1, \dots, n$  and  $k' \in \mathbb{N}$ . Hence we see  $\{e_k : k \in \mathbb{N}\} = \{s_J e_1 : J \in \mathcal{I}\}$ . From this and (B.1),  $e_1$  is a cyclic vector of  $\ell^2(\mathbb{N})$ . Let  $v = \sum_{m \geq 1} v_m e_m \in \ell^2(\mathbb{N})$ ,  $v \neq 0$ . Define  $m_0 = \min\{m \in \mathbb{N} : v_m \neq 0\}$ . Then there exists  $J_0 \in \mathcal{I}_1^k$  for some  $k \geq 1$  such that  $e_{m_0} = s_{J_0} e_1$ . Hence  $\langle e_1 | s_{J_0}^* v \rangle = v_{m_0} \neq 0$ . Therefore we can construct  $v' \in \mathcal{O}_n v$  such that  $v' = e_1 + v''$  for some  $v'' \in \ell^2(\mathbb{N})$ ,  $\langle e_1 | v'' \rangle = 0$ . Then we can verify  $\|(s_1^*)^l v' - e_1\| \rightarrow 0$  for  $l \rightarrow \infty$ . Therefore  $e_1 \in \overline{\mathcal{O}_n v}$ . This implies that any non-zero invariant closed subspace of  $\ell^2(\mathbb{N})$  coincides with  $\ell^2(\mathbb{N})$ . Therefore the action in (B.1) is irreducible. Hence  $\omega_1$  is pure.

Assume  $n = \infty$ . Define the action of  $\mathcal{O}_\infty$  on  $\ell^2(\mathbb{N})$  by

$$s_i e_k := e_{2^{i-1}(2k-1)} \quad (i, k \geq 1). \quad (\text{B.2})$$

Then  $s_1 e_1 = e_1$  and  $\{s_J e_k : J\} = \{e_m : m \in \mathbb{N}\}$ . Therefore  $e_1$  is a cyclic unit vector of  $\ell^2(\mathbb{N})$  and  $\omega_1 := \langle e_1 | (\cdot) e_1 \rangle$  satisfies  $\omega_1(s_1) = 1$ . As the same token, we can prove that the action in (B.2) is irreducible. Hence  $\omega_1$  is pure. (ii) Let  $U(\mathfrak{h})$  denote the unitary group on  $\mathfrak{h}$ . Let  $\{e_i\}$  denote the standard basis of  $\mathfrak{h}$ . For  $z \in \mathfrak{h}_1$ , let  $g = (g_{ij}) \in U(\mathfrak{h})$  such that  $g e_1 = z$  where  $g_{ij} := \langle e_i | g e_j \rangle$ . Then  $g_{j1} = z_j$  for all  $j$ . For  $\omega_1$  in (i), define

$$\omega' := \omega_1 \circ \alpha_{g^*} \quad (\text{B.3})$$

where  $\alpha$  is as in (1.8), which can be also well defined when  $n = \infty$ . By (B.3),  $\omega'$  is pure and we can verify  $\omega'(s_j) = \overline{z_j}$  for all  $j$  where we use  $\omega_1(s_i) = 0$  when  $i \neq 1$ . Hence  $\omega'$  is a Cuntz state by  $z$ . Therefore the existence is proved.

If  $\omega''$  is a Cuntz state by  $z$ , then we can verify that  $(\omega'' \circ \alpha_g)(s_1) = 1$  for  $g$  in (B.3). This implies that  $\omega'' \circ \alpha_g = \omega_1$  in (i) and  $\omega'' = \omega_1 \circ \alpha_{g^*} = \omega'$ . Hence the uniqueness of the Cuntz state by  $z$  is proved.

(iii) Assume  $\omega_z \sim \omega_y$ . Then there exists an action of  $\mathcal{O}_n$  on a Hilbert space  $\mathcal{H}$  with two cyclic unit vectors  $\Omega_z$  and  $\Omega_y$  such that  $s(z)\Omega_z = \Omega_z$  and  $s(y)\Omega_y = \Omega_y$  from Lemma 3.2 (the case of  $n = \infty$  also holds) where  $s(z) := \sum z_j s_j \in \mathcal{O}_n$ . Then  $\langle \Omega_z | s_J \Omega_y \rangle = \overline{z_J} \langle \Omega_z | \Omega_y \rangle$  for any  $J \in \mathcal{I}_1^k$ . Since  $\{s_J \Omega_y : J\}$  spans  $\mathcal{H}$ ,  $\langle \Omega_z | \Omega_y \rangle \neq 0$  because  $\Omega_z \neq 0$ . On the other hand,

$$\langle \Omega_z | \Omega_y \rangle = \langle s(z)\Omega_z | s(y)\Omega_y \rangle = \langle \Omega_z | s(z)^* s(y)\Omega_y \rangle = \langle z | y \rangle \langle \Omega_z | \Omega_y \rangle. \quad (\text{B.4})$$

Since  $\langle \Omega_z | \Omega_y \rangle \neq 0$ ,  $\langle z | y \rangle$  must be 1. This implies  $z = y$  from Lemma A.1. The inverse direction is trivial.

(iv) By definition, the statement is verified. ■

## C Sub-Cuntz states on $\mathcal{O}_\infty$

In this section, we generalize sub-Cuntz states on  $\mathcal{O}_n$  ( $n < \infty$ ) to  $\mathcal{O}_\infty$ . Except some parts, main theorems and properties of the state parametrization hold like the case of  $n < \infty$ . Hence we list different points and some remarks for the case of  $\mathcal{O}_\infty$ .

### C.1 Definition and parametrization

Let  $\mathbb{N} := \{1, 2, \dots\}$  and let  $\mathcal{O}_\infty$  denote the *Cuntz algebra* [12], that is, a  $C^*$ -algebra which is universally generated by  $\{s_i : i \in \mathbb{N}\}$  satisfying

$$s_i^* s_j = \delta_{ij} I \quad (i, j \in \mathbb{N}), \quad \sum_{i=1}^k s_i s_i^* \leq I \quad \text{for any } k \in \mathbb{N}. \quad (\text{C.1})$$

For a unit vector  $z \in \ell^2(\mathbb{N})$ ,  $\omega_z$  is a *Cuntz state* on  $\mathcal{O}_\infty$  by  $z$  if  $\omega_z$  is a state on  $\mathcal{O}_\infty$  which satisfies  $\omega_z(s_i) = \bar{z}_i$  for all  $i$ . Then  $\omega_z$  exists uniquely and is pure for any  $z$ ;  $\omega_z \sim \omega_y$  if and only if  $z = y$  (see Appendix B).

**Theorem C.1** *For  $m \geq 1$  and a unit vector  $z = \sum z_J e_J \in \ell^2(\mathbb{N}^m)$ , there exists a state  $\omega$  on  $\mathcal{O}_\infty$  which satisfies  $\omega(s_J) = \bar{z}_J$  for all  $J \in \mathbb{N}^m$ . Such  $\omega$  is called a sub-Cuntz state by  $z$  of order  $m$ .*

*Proof.* Fix a bijection  $f : \mathbb{N} \cong \mathbb{N}^m$  and define the endomorphism  $\hat{f}$  of  $\mathcal{O}_\infty$  by  $s_i \mapsto \hat{f}(s_i) := s_{f(i)}$  for each  $i \in \mathbb{N}$  where  $s_{f(i)} := s_{j_1} \cdots s_{j_m}$  when  $f(i) = (j_1, \dots, j_m)$ . Then  $\hat{f}(\mathcal{O}_\infty) \cong \mathcal{O}_\infty$  and  $\hat{f}(s_i)$ 's are Cuntz generators of  $\hat{f}(\mathcal{O}_\infty)$ . Then, for a unit vector  $z \in \ell^2(\mathbb{N}^m)$ ,  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_\infty$  by  $z$  if and only if  $\omega$  is an extension of the Cuntz state  $\omega_{\hat{z}}$  on  $\hat{f}(\mathcal{O}_\infty)$  by  $\hat{z}$  to  $\mathcal{O}_\infty$  where  $\hat{z} := (z_{f(i)})_{i \in \mathbb{N}} \in \ell^2(\mathbb{N})$ . Since an extension of  $\omega_{\hat{z}}$  to  $\mathcal{O}_\infty$  always exists from Proposition 2.1(i), the statement holds.  $\blacksquare$

Let  $\mathcal{V}_{\infty, m} := \ell^2(\mathbb{N}^m)$  with the standard basis  $\{e_J : J \in \mathbb{N}^m\}$ . We identify  $\mathcal{V}_{\infty, m}$  with  $\ell^2(\mathbb{N})^{\otimes m}$ . Then the periodicity, decomposability and equivalence of parameters are defined as same as the case of  $n < \infty$ .

Let  $\omega$  be a state on  $\mathcal{O}_\infty$  with the GNS representation  $(\mathcal{H}, \pi, \Omega)$  and  $z = \sum z_J e_J \in (\ell^2(\mathbb{N}^m))_1$ . In a similar fashion with Theorem 2.3, we see that the following are equivalent: (i)  $\omega$  is a sub-Cuntz state by  $z$ . (ii)  $\sum z_J \pi(s_J) \Omega = \Omega$ . (iii)  $\pi(s_J)^* \Omega = z_J \Omega$  for all  $J$ . Remark that  $s(z) = \sum z_J s_J$  is well defined in  $\mathcal{O}_\infty$  for any  $z$ . Hence the l.h.s in (ii) is well defined.

## C.2 Main theorems and naturalities of parametrization

Statements in main theorems are almost same with the case of  $n < \infty$ . Let  $U(\infty)$  denote the group of all unitaries in  $\mathcal{B}(\ell^2(\mathbb{N}))$ . Then the state parametrization  $z \mapsto \tilde{\omega}_z$  satisfies the  $U(\infty)$ -covariance.

For the  $\varphi$ -tensor product of states on  $\mathcal{O}_\infty$ , the following new definitions are necessary. For  $2 \leq n < \infty$ , let  $\{s_k^{(\infty)}\}$  and  $\{s_i^{(n)}\}$  denote Cuntz generators of  $\mathcal{O}_\infty$  and  $\mathcal{O}_n$ , respectively. Define the embedding  $\varphi_{\infty,n}$  of  $\mathcal{O}_\infty$  into  $\mathcal{O}_\infty \otimes \mathcal{O}_n$  by  $\varphi_{\infty,n}(s_{n(k-1)+i}^{(\infty)}) := s_k^{(\infty)} \otimes s_i^{(n)}$  for  $k \in \mathbb{N}$ ,  $i = 1, \dots, n$ . For  $\omega_1 \in \mathcal{S}_\infty$  and  $\omega_2 \in \mathcal{S}_n$ , define  $\omega_1 \otimes_\varphi \omega_2 \in \mathcal{S}_\infty$  by  $\omega_1 \otimes_\varphi \omega_2 := (\omega_1 \otimes \omega_2) \circ \varphi_{\infty,n}$ . For  $\omega_1 \in \mathcal{S}_\infty$ ,  $\omega_2 \in \mathcal{S}_n$  and  $\omega_3 \in \mathcal{S}_{n'}$ ,  $(\omega_1 \otimes_\varphi \omega_2) \otimes_\varphi \omega_3 = \omega_1 \otimes_\varphi (\omega_2 \otimes_\varphi \omega_3)$  ([35], the proof of Theorem 1.2(iv)). If  $\omega_1$  and  $\omega_2$  are sub-Cuntz states on  $\mathcal{O}_\infty$  and  $\mathcal{O}_n$ , respectively, then we see that  $\omega_1 \otimes_\varphi \omega_2$  is a sub-Cuntz state on  $\mathcal{O}_\infty$ .

For  $J = (j_1, \dots, j_m) \in \mathbb{N}^m$  and  $K = (k_1, \dots, k_m) \in \{1, \dots, n\}^m$ , define  $J \boxtimes K = (l_1, \dots, l_m) \in \mathbb{N}^m$  by  $l_t := n(j_t - 1) + k_t$  for  $t = 1, \dots, m$ . For  $z \in \mathcal{V}_{\infty,m}$  and  $y \in \mathcal{V}_{n,m}$ , define  $z \boxtimes y \in \mathcal{V}_{\infty,m}$  by

$$z \boxtimes y := \sum_{J \in \mathbb{N}^m} (z \boxtimes y)_{J e_J}, \quad (z \boxtimes y)_J := z_{J'} y_{J''} \quad (\text{C.2})$$

where  $J' \in \mathbb{N}^m$  and  $J'' \in \{1, \dots, n\}^m$  are uniquely defined as  $J = J' \boxtimes J''$ . For  $z \in \mathcal{V}_{\infty,m}$  and  $y \in \mathcal{V}_{n,l}$ , define  $z * y \in \mathcal{V}_{\infty,\alpha m}$  by  $z * y := z^{\otimes \alpha} \boxtimes y^{\otimes \beta}$  where  $\alpha$  and  $\beta$  are chosen such that  $\alpha m = \beta l$  is the least common multiple of  $m$  and  $l$ . If  $z \in \mathcal{V}_{\infty,m}$  and  $y \in \mathcal{V}_{n,l}$  are nonperiodic, then  $z * y$  is also nonperiodic and  $\tilde{\omega}_z \otimes_\varphi \tilde{\omega}_y = \tilde{\omega}_{z*y}$ .

## C.3 Infinitely correlated sub-Cuntz states on $\mathcal{O}_\infty$

Lemma 2.4(i) does not hold for  $\mathcal{O}_\infty$ . We prove that a sub-Cuntz state on  $\mathcal{O}_\infty$  is not always finitely correlated by using examples.

**Proposition C.2** *For a unit vector  $x = \sum x_i e_i \in \ell^2(\mathbb{N})$ , define  $z := \sum x_j e_j \otimes e_j \in (\ell^2(\mathbb{N})^{\otimes 2})_1$ ,  $X := \{i \in \mathbb{N} : x_i \neq 0\}$  and  $N := \#X$ . Assume that  $\omega$  is a sub-Cuntz state on  $\mathcal{O}_\infty$  by  $z$ . Then  $\omega$  is finitely correlated if and only if  $N < \infty$ .*

*Proof.* Let  $(\mathcal{H}, \pi, \Omega)$  denote the GNS representation by  $\omega$ . By definition,  $\omega(s_i s_j) = \delta_{ij} \overline{x_i} x_j$  for all  $i, j$ . This implies  $\sum x_i \pi(s_i^2) \Omega = \Omega$  and

$$\omega(s_i s_j^*) = \sum_l \delta_{il} \delta_{jl} \overline{x_i} x_j = \delta_{ij} |x_i|^2 \quad \text{for all } i, j. \quad (\text{C.3})$$

We divide the case of  $N \geq 2$  from the case of  $N = 1$ .

Assume  $N \geq 2$ . In this case,  $z$  is nonperiodic. From Theorem 1.4(iii) for  $\mathcal{O}_\infty$ ,  $\langle \Omega | \pi(s_i)^* \Omega \rangle = \omega(s_i^*) = 0$  for all  $i$ . From this and (C.3),  $\{\Omega, |x_i|^{-1} \pi(s_i^*) \Omega : i \in X\}_{i \geq 1}$  is an orthonormal family in  $\mathcal{H}$ . Define  $\mathcal{I} := \bigcup_{a \geq 0} \mathbb{N}^a$ . Then  $\mathcal{I}$  is a free monoid with respect to the concatenation [47]. Define the subsemigroup  $W$  of  $\mathcal{I}$  generated by  $\{(ii) : i \in \mathbb{N}\}$ :

$$W := \langle \{(ii) : i \in \mathbb{N}\} \rangle \subset \mathcal{I}. \quad (\text{C.4})$$

Then

$$\pi(s_J^*) \Omega = \begin{cases} x[J_1] \pi(s_i)^* \Omega & \text{when } J = J_1 i, J_1 \in W, \\ x[J] \Omega & \text{when } J \in W, \\ 0 & \text{otherwise} \end{cases} \quad (\text{C.5})$$

where  $x[J] := x_{i_1} x_{i_2} \cdots x_{i_l} \in \mathbb{C}$  when  $J = (i_1 i_1 i_2 i_2 \cdots i_l i_l) \in W$ . From this,  $\text{Lin}\langle \{\pi(s_J)^* \Omega : J\} \rangle = \text{Lin}\langle \{\Omega, \pi(s_i)^* \Omega : i \in X\} \rangle$ . Therefore

$$\dim \mathcal{K} = \dim \text{Lin}\langle \{\Omega, \pi(s_i)^* \Omega : i \in X\} \rangle = 1 + \#X = 1 + N \quad (\text{C.6})$$

where  $\mathcal{K} := \overline{\text{Lin}\langle \{\pi(s_J)^* \Omega : J\} \rangle} \subset \mathcal{H}$ . From (C.6), the statement holds except the case of  $N = 1$ .

Assume  $N = 1$ . It is sufficient to show that  $\omega$  is finitely correlated. By assumption, there exists  $j$  such that  $z = x_j e_j \otimes e_j$  and  $|x_j| = 1$ . In this case, we obtain  $x_j \pi(s_j^2) \Omega = \Omega$ . From this,

$$\pi(s_i s_k)^* \Omega = \delta_{ij} \delta_{jk} x_j \Omega \quad \text{for all } i, k. \quad (\text{C.7})$$

Let  $q$  be a quadratic root of  $x_j$ . Then there exists  $0 \leq a \leq 1$  such that  $\omega = a\omega_+ + (1-a)\omega_-$  where  $\omega_\pm$  denotes the Cuntz state by  $\pm q e_j$ . In the proof of Theorem 1.5, there exists a unit vector  $(\alpha, \beta) \in \mathbb{R}^2$  such that  $\Omega = \alpha\Omega_+ + \beta\Omega_-$  and  $\alpha^2 = a$  and  $\beta^2 = 1-a$  where  $\Omega_\pm$  denotes the GNS cyclic vector by  $\omega_\pm$ . Then we see that  $\pi(s_i)^* \Omega = q\delta_{ij}(\alpha\Omega_+ - \beta\Omega_-)$  for all  $i$ . From this and (C.7),  $\pi(s_J)^* \Omega \in \text{Lin}\langle \{\Omega, \alpha\Omega_+ - \beta\Omega_-\} \rangle$  for any  $J$ . Therefore  $\dim \mathcal{K} \leq \dim \text{Lin}\langle \{\Omega, \alpha\Omega_+ - \beta\Omega_-\} \rangle \leq 2$ . Hence  $\omega$  is finitely correlated. ■

For example, let  $x := \sum 2^{-i/2} e_i \in (\ell^2(\mathbb{N}))_1$ . Then  $\omega$  associated with  $x$  satisfies  $N = \infty$  in Proposition C.2.

## References

- [1] M. Abe, K. Kawamura, Pseudo-Cuntz algebra and recursive FP ghost system in string theory, Int. J. Mod. Phys. A 18(4) (2003), 607–625.

- [2] M. Abe, K. Kawamura, Branching laws for endomorphisms of fermions and the Cuntz algebra  $\mathcal{O}_2$ , J. Math. Phys. 49 (2008), 043501-01–043501-10.
- [3] H. Araki, A.L. Carey, D.E. Evans, On  $\mathcal{O}_{n+1}$ , J. Operator Theory 12 (1984), 247–264.
- [4] W.R. Bergmann, R. Conti, Induced product representation of extended Cuntz algebras, Annali di Matematica 182 (2003), 271–286.
- [5] R. Bhatia, Matrix analysis, Springer, 1997.
- [6] O. Bratteli, P.E.T. Jorgensen, Endomorphisms of  $\mathcal{B}(\mathcal{H})$  II. Finitely correlated states on  $\mathcal{O}_n$ , J. Funct. Anal. 145 (1997), 323–373.
- [7] O. Bratteli, P.E.T. Jorgensen, Iterated function systems and permutation representations of the Cuntz algebra, Mem. Amer. Math. Soc. 139 (1999), 1–89.
- [8] O. Bratteli, P.E.T. Jorgensen, A. Kishimoto, R.F. Werner, Pure states on  $\mathcal{O}_d$ , J. Operator Theory 43(1) (2000), 97–143.
- [9] O. Bratteli, P.E.T. Jorgensen, V. Ostrovskyĭ, Representation theory and numerical AF-invariants. The representations and centralizers of certain states on  $\mathcal{O}_d$ , Mem. Amer. Math. Soc. 168(797) (2004), 1–178.
- [10] O. Bratteli, P.E.T. Jorgensen, G.L. Price, Endomorphisms of  $\mathcal{B}(\mathcal{H})$ , in Quantization, nonlinear partial differential equations, and operator algebra (W. Arveson, T. Branson, and I. Segal, eds.), Proc. Sympos. Pure Math., 59, Amer. Math. Soc., 1996, pp. 93–138.
- [11] A.H. Clifford, G.B. Preston, The algebraic theory of semigroup vol. II, American Mathematical Society, 1967.
- [12] J. Cuntz, Simple  $C^*$ -algebras generated by isometries, Commun. Math. Phys. 57 (1977), 173–185.
- [13] K.R. Davidson, D.R. Pitts, The algebraic structure of non-commutative analytic Toeplitz algebras, Math. Ann. 311 (1998), 275–303.
- [14] K.R. Davidson, D.R. Pitts, Invariant subspaces and hyper-reflexivity for free semigroup algebras, Proc. London Math. Soc. 78 (1999), 401–430.
- [15] J. Dixmier,  $C^*$ -algebras, North-Holland Publishing Company, 1977.

- [16] N. Dunford, J.T. Schwartz, Linear operators. II, Interscience, New York, 1963.
- [17] D.E. Dutkay, J. Haussermann, P.E.T. Jorgensen, Atomic representations of Cuntz algebras, arXiv:1311.5265v1.
- [18] L. Eldén, B. Savas, A Newton-Grassmann method for computing the best multilinear rank- $(r_1, r_2, r_3)$  approximation of a tensor, SIAM. J. Matrix Anal. & Appl., 31(2) (2009), 248–271.
- [19] D.E. Evans, On  $\mathcal{O}_n$ , Publ. RIMS, Kyoto Univ. 16 (1980), 915–927.
- [20] N.J. Fowler, M. Laca, Endomorphisms of  $\mathcal{B}(\mathcal{H})$ , extensions of pure states, and a class of representations of  $\mathcal{O}_n$ , J. Operator Theory 40(1) (2000), 113–138.
- [21] M.J. Gabriel, Cuntz algebra states defined by implementers of endomorphisms of the CAR algebra, Canad. J. Math. 54 (2002), 694–708.
- [22] J. Glimm, Type I  $C^*$ -algebras, Ann. Math. 73(3) (1961), 572–612.
- [23] G.H. Golub, C.F. Van, Matrix computations 3rd ed., The Johns Hopkins University Press, 1996.
- [24] J.M. Howie, Fundamentals of semigroup theory, Oxford Science Publications, 1995.
- [25] M. Izumi, Subalgebras of infinite  $C^*$ -algebras with finite Watatani indices. I. Cuntz algebras, Commun. Math. Phys. 155(1) (1993), 157–182.
- [26] E.-C. Jeong, Irreducible representations of the Cuntz algebra  $\mathcal{O}_n$ , Proc. Amer. Math. Soc. 127(12) (1999), 3583–3590.
- [27] E.-C. Jeong, Linear functionals on the Cuntz algebra, Proc. Amer. Math. Soc. 134(1) (2005), 99–104.
- [28] K. Kawamura, Generalized permutative representations of the Cuntz algebras, arXiv:math/0505101.
- [29] K. Kawamura, Extensions of representations of the CAR algebra to the Cuntz algebra  $\mathcal{O}_2$  —the Fock and the infinite wedge—, J. Math. Phys. 46(7) (2005), 073509-1–073509-12.
- [30] K. Kawamura, The Perron-Frobenius operators, invariant measures and representations of the Cuntz-Krieger algebras, J. Math. Phys. 46(8) (2005), 083514-1–083514-6.

- [31] K. Kawamura, Polynomial endomorphisms of the Cuntz algebras arising from permutations. I —General theory—, *Lett. Math. Phys.* 71 (2005), 149–158.
- [32] K. Kawamura, Branching laws for polynomial endomorphisms of Cuntz algebras arising from permutations, *Lett. Math. Phys.* 77 (2006), 111–126.
- [33] K. Kawamura, A tensor product of representations of Cuntz algebras, *Lett. Math. Phys.* 82(1) (2007), 91–104.
- [34] K. Kawamura, Automata computation of branching laws for endomorphisms of Cuntz algebras, *Int. J. Alg. Comput.* 17(7) (2007), 1389–1409.
- [35] K. Kawamura,  $C^*$ -bialgebra defined by the direct sum of Cuntz algebras, *J. Algebra* 319 (2008), 3935–3959.
- [36] K. Kawamura, Classification and realizations of type III factor representations of Cuntz-Krieger algebras associated with quasi-free states, *Lett. Math. Phys.* 87 (2009), 199–207.
- [37] K. Kawamura, Pentagon equation arising from state equations of a  $C^*$ -bialgebra, *Lett. Math. Phys.* 93 (2010), 229–241.
- [38] K. Kawamura,  $R$ -matrices and the Yang-Baxter equation on GNS representations of  $C^*$ -bialgebras, *Linear Alg. Appl.* 438 (2013), 573–583.
- [39] K. Kawamura, Tensor products of type III factor representations of Cuntz-Krieger algebras, *Algebr. Represent. Theor.* 16(5) (2013), 1397–1407.
- [40] K. Kawamura, Y. Hayashi, D. Lascu, Continued fraction expansions and permutative representations of the Cuntz algebra  $\mathcal{O}_\infty$ , *J. Number Theory* 129 (2009), 3069–3080.
- [41] T. Kobayashi, Theory of discretely decomposable restrictions of unitary representations of semisimple Lie groups and some applications, (translation of *Sūgaku* 51(4) (1999), 337–356), *Sugaku Expositions* 18(1) (2005), 1–37.
- [42] M. Laca, Gauge invariant states of  $\mathcal{O}_\infty$ , *J. Operator Theory* 30(2) (1993), 381–396.
- [43] S. Lang, *Algebra Revised 3rd ed.*, Springer, 2002.

- [44] L.D. Lathauwer, B.D. Moor, J. Vandewalle, A multilinear singular value decomposition, *SIAM J. Matrix Anal. Appl.*, 21 (2000), 1253–1278.
- [45] M.V. Lawson, Primitive partial permutation representations of the polycyclic monoids and branching function systems, *Periodica Mathematica Hungarica* 58 (2) (2009), 189–207.
- [46] J.-R. Lee, D.-Y. Shin, The positivity of linear functionals on Cuntz algebras associated to unit vectors, *Proc. Amer. Math. Soc.* 132(7) (2004), 2115–2119.
- [47] M. Lothaire, *Combinatorics on words*, Cambridge University Press, 1983.
- [48] D.-Y. Shin, State extensions of states on  $\mathrm{UHF}_n$  algebra to Cuntz algebra, *Bull. Korean Math. Soc.* 39(3) (2002), 471–478.
- [49] M.A.O. Vasilescu, D. Terzopoulos, Multilinear analysis of image ensembles: Tensorfaces. In *Proc. 7th European Conference on Computer Vision (ECCV'02)*, Lecture Notes in Computer Science, 2350 (2002), 447–460.